MAXWELL EQUATIONS IN MEDIA, GROUP THEORY AND POLARIZATION OF THE LIGHT

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Matrix form of Maxwell electrodynamics in media in the Riemann – Silberstein – Majorana – Oppenheimer approach requires two sets of commuting 4×4 matrices, α^b and β^b – simple and symmetrical realization of which is given. These matrices are used as a tool to construct parametrization of all 4×4 matrices, which in turn are considered in the context of polarization optics as Mueller matrices; a subset of them consists of a group isomorphic to Special Relativity Lorentz group $L^{\uparrow}_{+-}$.

In the paper a factorized technique of working with the Lorentz, which closely related to studies of Einstein and Mayer [1, 2, 3] on semi-vectors, and to the systematic construction of the Lorentz group theory on this base given by Fedorov [4], and also closely related to quaternionic approach [5, 6, 7], is used. This technique is specified for looking at the problems of light polarization optics in the frames of vector Stokes-Mueller and spinor Jones formalism. Translation to isotropic formalism by Newman and Penrose is described.

Some unsolved problems are discussed. Because Jones complex formalism has close relation to spinor objects of the Lorentz group, within the field of the light polarization we could have physical realizations on the optical desk of some subtle topological distinctions between orthogonal $L^{\uparrow}_{+} = SO_0(3.1)$ and spinor $SL(2.C)$ groups. These topological differences of the groups find their corollaries in the problem of the so-called spinor structure of physical space-time – the problem was extensively discussed in the literature.

Keywords: Maxwell equation. Lorentz group, light polarization, Mueller and Jones formalisms, spinor representation of Stokes 4-vectors, Newman – Penrose isotropic formalism, spinor space-time structure.

THE CONTENT

1 Introduction

There exist many different ways to approach the Lorentz group symmetry, the Lorentz group $L_+^\uparrow$; see [8] and references therein. Though historically this group arose within the physical problems of Maxwell electrodynamics of moving bodies (Lorentz, Poincaré, Einstein), nowadays special relativity exposition mainly is based on a logical line suggested by Einstein in 1905.

The aim the present article is to give a formal construction of the Lorentz group theory with the use of constructs arising within the electrodynamic Maxwell equations. Partly, the paper is written with a pedagogical view, the main claim (of course, not new one) is that Maxwell equations contain in themselves all mathematical tools to explore such topics as rotation and Lorentz group symmetry. Thereby, we have returned to the very beginning of relativistic symmetry and used the Maxwell equations to build the theory of the Lorentz group. We have started with a matrix form of Maxwell electrodynamics in media in the Riemann – Silberstein – Majorana – Oppenheimer approach (see [9] and references therein) which requires two sets of real and commuting 4×4 matrices, α^b and β^b. These matrices are used as a tool to construct the theory of the complex rotation group $O(4.C)$ on the base of the factorized relationship

$$R_{\alpha\beta} = (A^a \alpha^b)\ (B^b \beta^b) = (A_0\ I + A_i \alpha^i)\ (B_0\ I + B_j \beta^j)\ . \tag{1}$$

where A^a, B^b are taken as independent parameters. The most interesting cases of real Lorentz group $L_+^\uparrow$, real $O(3.R)$ and complex $O(3.C)$ rotation groups, and their spinor coverings, $SL(2.C)$ and $SU(2)$, are detailed. Translation to isotropic formalism [11] by Newman and Penrose is described.

To avoid misunderstanding it should note that for Maxwell theory maximal symmetry group coincides of cause with more complex conformal group $SO(4.2)$ (or its covering $SU(2.2)$), but for present exploration into polarization optics problems this more extended group does not seem to be necessary now.

The technique developed is specified for looking at the problems of light polarization in the frames of vector Stokes-Mueller and spinor Jones formalism. The main line of evolution in theoretical methods of polarization optics seems to be quite independent of that in relativistic symmetry methods, developed, for example, in particle physics. By many authors it was noticed that these two branches of physics employ, in fact, the same mathematical technique[1] , only with occasionally motivated distinctions in notation and physical accents. So, in this part, the present article is one other appeal: instead of remaking the same things in different embodiment, it is better to work out and adopt a unique mathematical language. In so doing we might see the more unity and simplicity in symmetry aspects of particle physics and optics of the polarized light. Because Jones complex formalism has close relation to spinor objects of the Lorentz group, within the field of the light polarization we could have physical realizations on the optical desk of some subtle topological distinctions between orthogonal $L_+^\uparrow = SO_0(3.1)$ and spinor $SL(2.C)$ groups. These topological differences of the groups find their corollaries in the problem of the so-called spinor structure of physical space-time.

Let us give several general remarks on aspects we are to focus.

Remembering on great differences between properties of isotropic and time-like vectors in Special Relativity we should expect the same principal differences in describing polarized and

[1]The bibliography on the subject is large enough, we have listed many of them; but the reader should remember on pedagogic missing of the paper and consider this list just as hints to further reading and thinking.

partly polarized light. So below we will be considering these two cases separately: a polarized light and a partly polarized light. Substantial differences will be revealed when turning to Lorentz boost transformations and to spinor techniques.

One may restrict oneself to rotation subgroup of the Lorentz group and study corresponding manifestation optical devices in such a non-relativistic limit. So, the main idea in the optical context should be to derive as much as possible from conventional theory of the rotation group $SO(3.R)$ and its spinor covering, unitary group $SU(2)$; turning to spinors of $SU(2)$ latter immediately leads us to an additional insight to complex Jones formalism in polarization optics.

We will describe some possibilities to apply in optical context the well-known and widely-spread formalism of Newman and Penrose [11], and closely related to it ideas on spinor space-time structure being reformulated to polarization optics.

2 Maxwell equations and mathematical structure of the Lorentz group

Let start with basic Maxwell equation in real vector form (in the system SI)

$$F^{ab} = (\mathbf{E},\ c\mathbf{B}), \qquad \text{div}\ \mathbf{B} = 0\ , \qquad \text{rot}\ \mathbf{E} = -\frac{\partial c\mathbf{B}}{\partial ct}\ ;$$
$$H^{ab} = (\mathbf{D},\ \mathbf{H}/c)\ , \qquad \text{div}\ \mathbf{D} = \rho\ , \qquad \text{rot}\ \frac{\mathbf{H}}{c} = \frac{\mathbf{J}}{c} + \frac{\partial \mathbf{D}}{\partial ct}\ ,$$

In terms of complex vectors under the orthogonal group SO(3.C) (for more detail see [9, 16]

$$\mathbf{f} = \mathbf{E} + ic\mathbf{B}\ , \qquad \mathbf{h} = \frac{1}{\epsilon_0}\ (\mathbf{D} + i\mathbf{H}/c)\ ,\ \ j^a = (j^0 = \rho,\ \mathbf{j} = \mathbf{J}/c)\ ; \tag{2}$$

Maxwell equations read (note that $x^0 = x_0 = ct$)

$$\text{div}\ (\frac{\mathbf{h}+\mathbf{h}^*}{2} + \frac{\mathbf{f}-\mathbf{f}^*}{2}) = \frac{1}{\epsilon_0}\ \rho\ ,$$
$$-i\partial_0(\frac{\mathbf{h}+\mathbf{h}^*}{2} + \frac{\mathbf{f}-\mathbf{f}^*}{2}) + \text{rot}\ (\frac{\mathbf{f}+\mathbf{f}^*}{2} + \frac{\mathbf{h}-\mathbf{h}^*}{2}) = \frac{i}{\epsilon_0}\ \mathbf{j}\ . \tag{3}$$

In new variables

$$\mathbf{M} = \frac{\mathbf{h}+\mathbf{f}}{2}\ , \qquad \mathbf{N} = \frac{\mathbf{h}^* - \mathbf{f}^*}{2}\ ,$$

eqs. (3) take the form

$$\text{div}\ (\mathbf{M}+\mathbf{N}) = \frac{1}{\epsilon_0}\ \rho\ ,$$
$$\partial_0(\mathbf{M}+\mathbf{N}) + i\ \text{rot}\ (\mathbf{M}-\mathbf{N}) = -\frac{1}{\epsilon_0}\ \mathbf{j}\ . \tag{4}$$

It should be stressed that $\mathbf{M}$ and $\mathbf{N}$ correspond respectively to different representations of the group SO(3.C):

$$\mathbf{M} = O(k)\ \mathbf{M}\ , \qquad \mathbf{N} = O^*(k)\ \mathbf{N}\ .$$

Equations (4) can be translated to matrix formalism (notation used according to [9, 16])

$$(-i\partial_0 + \alpha^i \partial_i)\, M + (-i\partial_0 + \beta^i \partial_i)\, N = J \;,$$

real matrices α^j, β^j were introduced which possess very simple properties:

$$(\alpha^1)^2 = -I \;, \qquad (\alpha^2)^2 = -I \;, \qquad (\alpha^3)^2 = -I \;,$$
$$\alpha^1\alpha^2 = -\alpha^2\alpha^1 = +\alpha^3 \;,\; \alpha^2\alpha^3 = -\alpha^3\alpha^2 = +\alpha^1 \;,\; \alpha^3\alpha^1 = -\alpha^1\alpha^3 = +\alpha^2 \;,$$
$$(\beta^1)^2 = -I \;, \qquad (\beta^2)^2 = -I \;, \qquad (\beta^3)^2 = -I \;,$$
$$\beta^1\beta^2 = -\beta^2\beta^1 = -\beta^3 \;,\; \beta^2\beta^3 = -\beta^3\beta^2 = -\beta^1 \;,\; \beta^3\beta^1 = -\beta^1\beta^3 = -\beta^2 \;;$$

besides, two sets commute with each other: $\alpha^k \beta^l = \beta^l \alpha^k$. Let us consider all pair products of I, α^k, β^k – there arise 16 matrices:

$$I,\; \alpha^1,\; \alpha^2,\; \alpha^3,\; \beta^1,\; \beta^2,\; \beta^3,$$
$$\alpha^1\beta^1,\; \alpha^1\beta^2,\; \alpha^1\beta^3,\; \alpha^2\beta^1,\; \alpha^2\beta^2,\; \alpha^2\beta^3,\; \alpha^3\beta^1,\; \alpha^3\beta^2,\; \alpha^3\beta^3 \;,$$

they represent a basis in 16-dimensional linear space of real (4×4) matrices:

$$R = E\, I + A_i\, \alpha^i + B_i\, \beta^i + C_{ij}\, \alpha^i \beta^j \;. \tag{5}$$

The formula (5) determines a special parametrization for real linear group $GL(4.R)$ and all its orthogonal and unitary subgroups; we get the complex linear group $GL(4.C)$ when using complex values for E, A_i, B_i, C_{ij} (see similar treatment on the base of the use of Dirac Matrices in [10]). Basis elements obey relationships:

$$\alpha^i \alpha^j = -\delta_{ij} + \epsilon_{ijk}\alpha_k \;, \qquad \beta^i\beta^j = -\delta_{ij} - \epsilon_{ijk}\beta_k \;, \qquad \alpha_i\beta_j = \beta_j\alpha_i;. \tag{6}$$

where

$$\alpha^1 = \begin{vmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{vmatrix} , \; \alpha^2 = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix} , \; \alpha^3 = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{vmatrix} ,$$

$$\beta^1 = \begin{vmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{vmatrix} , \; \beta^2 = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix} , \; \beta^3 = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{vmatrix} , \; .$$

One can easily derive the group composition rule $R'R$:

$$R'' = (E'\; I + A'_i\alpha^i + B'_i\beta^i + C'_{ij}\alpha^i\beta^j)\; (E\; I + A_j\alpha^j + B_j\beta^j + C_{kl}\alpha^k\beta^l)$$

for such parameters:

$$\begin{aligned} E'' &= E'\; E - A'_i\; A_i - B'_i\; B_i + C'_{ij}\; C_{ij} \;, \\ A''_n &= E'\; A_n + A'_n\; E + (\epsilon_{ijn}\; A'_i\; A_j) - \\ &\quad - C_{ni}\; B'_i - C'_{nl}\; B_l - C'_{ij}\; C_{kj}\;\; \epsilon_{ikn} \;, \\ B''_n &= E'\; B_n + B'_n\; E - (\epsilon_{nij}\; B'_i\; B_j) - \\ &\quad - A'_i\; C_{in} - A_i\; C'_{in} + C'_{ij}\; C_{il}\; \epsilon_{jln} \;, \\ C''_{nm} &= E'\; C_{nm} + C'_{nm}\; E + A'_n\; B_m + B'_n\; A_m - \\ &\quad - (A'_i\; \epsilon_{ink})\; C_{km} - \; C_{nl}\; (B'_i\; \epsilon_{ilm}) \; + \\ &\quad + (A_l\; \epsilon_{lni})\; C'_{im} + \; C'_{nj}\; (B_l\; \epsilon_{ljm}) - C'_{ij}\; C_{kl}\; \epsilon_{ikn}\; \epsilon_{jlm} \;. \end{aligned} \tag{7}$$

In the following we restrict ourselves to orthogonal subgroups (in this connection also see [8] and big list of references therein); for treatment of unitary and pseudo-unitary subgroups $SU(4), S(3.1), SU(2.2)$ see [10] (the group $SU(2.2)$ is spinor covering for 15-parametric conformal group being of primary significance for Maxwell theory; but for present exploration into polarization optics problems this more extended group does not seem to be necessary now).

Evidently, relations (7) allow two subgroups with simple composition rules:

$$R_\alpha(A_0, A_i) = A_0\, I + A_i \alpha^i = \left| \begin{array}{rrrr} A_0 & A_1 & A_2 & A_3 \\ -A_1 & A_0 & -A_3 & A_2 \\ -A_2 & A_3 & A_0 & -A_1 \\ -A_3 & -A_2 & A_1 & A_0 \end{array} \right| ,$$

$$A''_0 = A'_0\, A_0 - A'_i\, A_i\ , \qquad A''_n = A'_0\, A_n + A'_n\, A_0 + \epsilon_{ijn}\, A'_i\, A_j\ ;$$

$$R_\beta(B_0, B_i) = B_0\, I + B_i \beta^i = \left| \begin{array}{rrrr} B_0 & B_1 & B_2 & B_3 \\ -B_1 & B_0 & B_3 & -B_2 \\ -B_2 & -B_3 & B_0 & B_1 \\ -B_3 & B_2 & -B_1 & B_0 \end{array} \right| ,$$

$$B''_0 = B'_0\, B_0 - B'_i\, B_i\ , \qquad B''_n = B'_0\, B_n + B'_n\, B_0 - \epsilon_{ijn}\, B'_i\, B_j\ . \tag{8}$$

Two subgroups are isomorphic, indeed, let us change the variables $\tilde{A}_0 = B_0, \tilde{A}_i = -B_i$, then

$$R_\beta(B_0, B_i) = A_0\, I - \tilde{A}_i \beta^i = \left| \begin{array}{rrrr} \tilde{A}_0 & -\tilde{A}_1 & -\tilde{A}_2 & -\tilde{A}_3 \\ \tilde{A}_1 & \tilde{A}_0 & -\tilde{A}_3 & \tilde{A}_2 \\ \tilde{A}_2 & \tilde{A}_3 & \tilde{A}_0 & -\tilde{A}_1 \\ \tilde{A}_3 & -\tilde{A}_2 & \tilde{A}_1 & \tilde{A}_0 \end{array} \right| ,$$

$$\tilde{A}''_0 = \tilde{A}'_0\, \tilde{A}_0 - \tilde{A}'_i\, \tilde{A}_i\ , \qquad \tilde{A}''_n = \tilde{A}'_0\, \tilde{A}_n + \tilde{A}'_n\, \tilde{A}_0 + \epsilon_{ijn}\, \tilde{A}'_i\, \tilde{A}_j\ , \tag{9}$$

which coincides with (8). Besides, we can readily verify identity

$$\Delta R_\alpha(B_0, -B_j)\Delta^{-1} = R_\beta(B_0, B_j)\ , \qquad \Delta = \left| \begin{array}{rrrr} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{array} \right| . \tag{10}$$

The 4-parametric subgroup $R_\alpha(A_0, A_i)$ is closely related with the unitary group $U(2)$ or linear group $GL(2.C)$ at real and complex parameters, respectively. Indeed, let us consider the matrix $R_\alpha(A_0, A_i)$ as a linear transformation in 4-space:

$$\begin{aligned} x'_0 &= +A_0 x_0 + A_1 x_1 + A_2 x_2 + A_3 x_3\ , \\ x'_1 &= -A_1 x_0 + A_0 x_1 - A_3 x_2 + A_2 x_3\ , \\ x'_2 &= -A_2 x_0 + A_3 x_1 + A_0 x_2 - A_1 x_3\ , \\ x'_3 &= -A_3 x_0 - A_2 x_1 + A_1 x_2 + A_0 x_3\ , \end{aligned} \tag{11}$$

from whence it follows

$$\begin{aligned} -x'_1 + i x'_2 &= (A_0 - iA_3)(-x_1 + i x_2) + (-iA_1 - A_2)(x_3 + i x_0)\ , \\ x'_3 + i x'_0 &= (-iA_1 + A_2)(-x_1 + i x_2) + (A_0 + iA_3)(x_3 + i x_0)\ . \end{aligned}$$

The later is easily recognized as the spinor transformation (of the unitary $U(2)$ or linear $GL(2.C)$ depending on values of (A_0, A_i)):

$$\xi = \frac{1}{\sqrt{2}} \left| \begin{array}{c} -x_1 + ix_2 \\ x_3 + ix_0 \end{array} \right| , \qquad \xi' = (A_0 I - i\sigma_j A_j)\ \xi \ . \tag{12}$$

Analogously, from (11) one can derive the transformation law for another spinor:

$$\begin{aligned} -\,x_0' - ix_3' &= (A_0 - iA_3)\ (-x_0 - ix_3) + (-iA_1 - A_2)\ (x_2 - ix_1)\ , \\ x_2' - ix_1' &= (-iA_1 + A_2)\ (-x_0 - ix_3) + (A_0 + iA_3)\ (x_2 - ix_1)\ , \end{aligned}$$

that is

$$\eta = \frac{1}{\sqrt{2}} \left| \begin{array}{c} -x_0 - ix_3 \\ x_2 - ix_1 \end{array} \right| , \qquad \eta' = (A_0 I - i\sigma_j A_j)\ \eta \ . \tag{13}$$

At real-valued parameters (A_0, A_i), coordinates (x_0, x_j) play the role of real Kustaanheimo-Stieffel variables [13]; spinors η and ξ are referred to each other as follows: $\eta = \sigma^2 \xi^*$.

Connection between four coordinates (x_0, x_j) and two corresponding spinors ξ, η is given by a liner transformation U:

$$\left| \begin{array}{c} \xi^1 \\ \xi^2 \\ \eta^1 \\ \eta^2 \end{array} \right| = \frac{1}{\sqrt{2}} \left| \begin{array}{cccc} 0 & -1 & i & 0 \\ i & 0 & 0 & 1 \\ -1 & 0 & 0 & -i \\ 0 & -i & 1 & 0 \end{array} \right| \left| \begin{array}{c} x_0 \\ x_1 \\ x_2 \\ x_3 \end{array} \right| , \tag{14}$$

inverse relation is

$$\left| \begin{array}{c} x_0 \\ x_1 \\ x_2 \\ x_3 \end{array} \right| = \frac{1}{\sqrt{2}} \left| \begin{array}{cccc} 0 & -i & -1 & 0 \\ -1 & 0 & 0 & i \\ -i & 0 & 0 & 1 \\ 0 & 1 & i & 0 \end{array} \right| \left| \begin{array}{c} \xi^1 \\ \xi^2 \\ \eta^1 \\ \eta^2 \end{array} \right| . \tag{15}$$

It is readily proved that the matrix $R_\alpha(A_0, A_i)$ reduces to a quasi-diagonal form by a similarity transformation:

$$U\ R_\alpha(A_0, A_i)\ U^{-1} = \left| \begin{array}{cc} A_0 I - i\sigma_j A_j & 0 \\ 0 & A_0 I - i\sigma_j A_j \end{array} \right| . \tag{16}$$

Therefore, the matrices $R_\alpha(A_0, A_i)$ are isomorphic to a couple of 2-component spinor representations (of the unitary group $U(2)$ or linear group $GL(2.C)$).

The case (8) can be considered in the same manner:

$$\begin{aligned} X_0' &= B_0 X_0 + B_1 X_1 + B_2 X_2 + B_3 X_3\ , \\ X_1' &= -B_1 X_0 + B_0 X_1 + B_3 X_2 - B_2 X_3\ , \\ X_2' &= -B_2 X_0 - B_3 X_1 + B_0 X_2 + B_1 X_3\ , \\ X_3' &= -B_3 X_0 + B_2 X_1 - B_1 X_2 + B_0 X_3\ , \end{aligned} \tag{17}$$

from whence it follows

$$
\begin{aligned}
-X_1' - iX_2' &= (B_0 - iB_3)\,(-X_1 - iX_2) + (B_2 - iB_1)\,(X_3 + iX_0)\,, \\
X_3' + iX_0' &= (-B_2 - iB_1)(-X_1 - iX_2) + (B_0 + iB_3)(X_3 + iX_0)\,,
\end{aligned}
\tag{18}
$$

that is

$$
\Sigma = \frac{1}{\sqrt{2}} \left| \begin{array}{c} -X_1 - iX_2 \\ X_3 + iX_0 \end{array} \right| , \qquad \Sigma' = (B_0 I - i\sigma_j B_j)\Sigma\,. \tag{19}
$$

Also, from (17) one can derive

$$
\begin{aligned}
-X_0' - iX_3' &= (B_0 - iB_3)\,(-X_0 - iX_3) + (B_2 - iB_1)\,(-X_2 - iX_1)\,, \\
-X_2' - iX_1' &= (-B_2 - iB_1)\,(-X_0 - iX_3) + (B_0 + iB_3)\,(-X_2 - iX_1)\,;
\end{aligned}
\tag{20}
$$

the later is the transformation law for the spinor

$$
H = \frac{1}{\sqrt{2}} \left| \begin{array}{c} -X_0 - iX_3 \\ -X_2 - iX_1 \end{array} \right| , \qquad H' = (B_0 I - i\sigma_j B_j)H\,. \tag{21}
$$

Connection between variables (X_0, X_j) and two spinors Σ, H is described by the transformation V:

$$
\left| \begin{array}{c} \Sigma^1 \\ \Sigma^2 \\ H^1 \\ H^2 \end{array} \right| = \frac{1}{\sqrt{2}} \left| \begin{array}{cccc} 0 & -1 & -i & 0 \\ i & 0 & 0 & 1 \\ -1 & 0 & 0 & -i \\ 0 & -i & -1 & 0 \end{array} \right| \left| \begin{array}{c} X_0 \\ X_1 \\ X_2 \\ X_3 \end{array} \right| ; \tag{22}
$$

inverse relation is

$$
\left| \begin{array}{c} X_0 \\ X_1 \\ X_2 \\ X_3 \end{array} \right| = \frac{1}{\sqrt{2}} \left| \begin{array}{cccc} 0 & -i & -1 & 0 \\ -1 & 0 & 0 & i \\ i & 0 & 0 & -1 \\ 0 & 1 & i & 0 \end{array} \right| \left| \begin{array}{c} \Sigma^1 \\ \Sigma^2 \\ H^1 \\ H^2 \end{array} \right| . \tag{23}
$$

In addition, one easily proves an identity:

$$
V\ R_\beta(B_0, B_i)\ V^{-1} = \left| \begin{array}{cc} B_0 I - i\sigma_j B_j & 0 \\ 0 & B_0 I - i\sigma_j B_j \end{array} \right| . \tag{24}
$$

Therefore, the matrices $R_\beta(A_0, A_i)$ are isomorphic to a couple of 2-component spinor representations (of the unitary group $U(2)$ or linear group $GL(2.C)$).

Because two sets of matrices, each making up a subgroup, commute

$$
R_\alpha(A_0, A_i) R_\beta(B_0, B_i) = R(B_0, B_i) R(A_0, A_i)\,,
$$

their product is again a subgroup:

$GL(2.C) \otimes GL(2.C)$

$$
\begin{aligned}
R(A, B) &= (A_0\ I + A_i \alpha^i)\ (B_0\ I + B_j \beta^j) = \\
&= A_0 B_0 + B_0 A_i\ \ \alpha^i + A_0 B_i\ \ \beta^i + A_i B_j\ \ \alpha^i \beta^j\,.
\end{aligned}
\tag{25}
$$

Within $R(A,B)$ one can separate a special subgroup by adding an additional constrain, $B_i = -A_i$:

$$R(A,\bar{A}) = R_\alpha(A_0,A_i) R_\beta(A_0,-A_i) =$$
$$= A_0 A_0 + A_0 A_i \ \ \alpha^i - A_0 A_i \ \ \beta^i - A_i A_j \ \ \alpha^i \beta^j \ . \qquad (26)$$

Explicit form of $R(A,\bar{A})$ is

$$R(A,\bar{A}) = R_\alpha(A_0,A_i) R_\beta(A_0,-A_i) =$$
$$= \left| \begin{array}{cccc} A^2 & 0 & 0 & 0 \\ 0 & A^2 - 2(A_2^2 + A_3^2) & -2A_0A_3 + 2A_1A_2 & 2A_0A_2 + 2A_1A_3 \\ 0 & 2A_0A_3 + 2A_1A_2 & A^2 - 2(A_1^2 + A_3^2) & -2A_0A_1 + 2A_2A_3 \\ 0 & -2A_0A_2 + 2A_1A_3 & 2A_0A_1 + 2A_2A_3 & A^2 - 2(A_1^2 + A_2^2) \end{array} \right| , \qquad (27)$$

where $A^2 = A_0^2 + A_1^2 + A_2^2 + A_3^2$. The determinant of the matrix $R(A,\bar{A})$ is given by

$$\det R(A,\bar{A}) = (A_0^2 + A_1^2 + A_2^2 + A_3^2)^4 \ .$$

Imposing restriction $\det R = 1$ (or $A^2 = 1$), we get

$$R = \left| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 - 2(A_2^2 + A_3^2) & -2A_0A_3 + 2A_1A_2 & 2A_0A_2 + 2A_1A_3 \\ 0 & 2A_0A_3 + 2A_1A_2 & 1 - 2(A_1^2 + A_3^2) & -2A_0A_1 + 2A_2A_3 \\ 0 & -2A_0A_2 + 2A_1A_3 & 2A_0A_1 + 2A_2A_3 & 1 - 2(A_1^2 + A_2^2) \end{array} \right| . \qquad (28)$$

Here, a (3×3) block is a matrix from complex orthogonal group $SO(3.C)$, or $SO(3.R)$ at real parameters.

In turn, for $R_{\alpha\beta}(A,B)$ in (25) we easily derive

$$R_{\alpha\beta} = R_\alpha(A_0,A_i) \ R_\beta(B_0,B_i) \ ,$$

$$\det \ (A_0 \ I + A_i \alpha^i) = (A_0^2 + A_1^2 + A_2^2 + A_3^2)^2 = A^4 \ ,$$
$$[R_\alpha(A_0,A_i)]^{-1} = \frac{1}{A^2} \ R_\alpha(A_0,-A_i) = \frac{1}{A^2} \ [R_\alpha(A_0,A_i)]^t \ ,$$

$$\det \ (B_0 \ I + B_i \alpha^i) = (B_0^2 + B_1^2 + B_2^2 + B_3^2)^2 = B^4 \ ,$$
$$[R_\alpha(B_0,B_i)]^{-1} = \frac{1}{B^2} \ R_\alpha(B_0,-B_i) = \frac{1}{B^2} \ [R_\alpha(B_0,B_i)]^t \ ; \qquad (29)$$

the symbol t stands for a matrix transposition. The property $R_{\alpha\beta} \in SO(4.C)$ is achieved by adding two constrains:

$$A^2 = A_0^2 + A_1^2 + A_2^2 + A_3^2 = A^2 = +1 \ ,$$
$$B^2 = B_0^2 + B_1^2 + B_2^2 + B_3^2 = B^2 = +1 \ . \qquad (30)$$

Let us find explicit form of the matrices $R_{\alpha\beta}$, it is convenient to write down their two parts

separately:

$$
A_0B_0 + B_0A_i\ \alpha^i + A_0B_i\ \beta^i =
$$

$$
\left| \begin{array}{cccc}
A_0B_0 & B_0A_1 + A_0B_1 & B_0A_2 + A_0B_2 & B_0A_3 + A_0B_3 \\
-B_0A_1 - A_0B_1 & A_0B_0 & -B_0A_3 + A_0B_3 & +B_0A_2 - A_0B_2 \\
-B_0A_2 - A_0B_2 & B_0A_3 - A_0B_3 & A_0B_0 & -B_0A_1 + A_0B_1 \\
-B_0A_3 - A_0B_3 & -B_0A_2 + A_0B_2 & +B_0A_1 - A_0B_1 & A_0B_0
\end{array} \right| ,
$$

$$
A_iB_j\ \ \alpha^i\beta^j =
$$

$$
\left| \begin{array}{rrrr}
-A_jB_j & A_3B_2 - A_2B_3 & A_1B_3 - A_3B_1 & A_2B_1 - A_1B_2 \\
-A_2B_3 + A_3B_2 & -A_1B_1 + A_2B_2 + A_3B_3 & -A_1B_2 - A_2B_1 & -A_1B_3 - A_3B_1 \\
A_1B_3 - A_3B_1 & -A_1B_2 - A_2B_1 & A_1B_1 - A_2B_2 + A_3B_3 & -A_2B_3 - A_3B_2 \\
-A_1B_2 + A_2B_1 & -A_1B_3 - A_3B_1 & -A_2B_3 - A_3B_2 & A_1B_1 + A_2B_2 - A_3B_3
\end{array} \right| .
$$

Expressions for matrices $R_{\alpha\beta}$ are rather cumbersome, they can be greatly simplified with the help of a special similarity transformation (see (14)):

$$
UR_{\alpha\beta}U^{-1} = (UR_\alpha U^{-1})\ (UR_\beta U^{-1})\ ; \tag{31}
$$

the first term is (new parameters a, b, c, d are introduced)

$$
UR_\alpha(A_0, A_i)U^{-1} = \left| \begin{array}{cccc}
A_0 - iA_3 & -iA_1 - A_2 & 0 & 0 \\
-iA_1 + A_2 & A_0 + iA_3 & 0 & 0 \\
0 & 0 & A_0 - iA_3 & -iA_1 - A_2 \\
0 & 0 & -iA_1 + A_2 & A_0 + iA_3
\end{array} \right| \equiv \left| \begin{array}{cccc}
a & d & 0 & 0 \\
c & b & 0 & 0 \\
0 & 0 & a & d \\
0 & 0 & c & b
\end{array} \right| , \tag{32}
$$

and the second term is (new parameters A, B, C, D are introduced as well)

$$
UR_\beta U^{-1} = \left| \begin{array}{cccc}
B_0 + iB_3 & 0 & -B_1 + iB_2 & 0 \\
0 & B_0 + iB_3 & 0 & -B_1 + iB_2 \\
B_1 + iB_2 & 0 & B_0 - iB_3 & 0 \\
0 & B_1 + iB_2 & 0 & B_0 - iB_3
\end{array} \right| = \left| \begin{array}{cccc}
A & 0 & D & 0 \\
0 & A & 0 & D \\
C & 0 & B & 0 \\
0 & C & 0 & B
\end{array} \right| ; \tag{33}
$$

and finally we arrive at

$$
UR_{\alpha\beta}U^{-1} = \left| \begin{array}{cccc}
a & d & 0 & 0 \\
c & b & 0 & 0 \\
0 & 0 & a & d \\
0 & 0 & c & b
\end{array} \right| \left| \begin{array}{cccc}
A & 0 & D & 0 \\
0 & A & 0 & D \\
C & 0 & B & 0 \\
0 & C & 0 & B
\end{array} \right| = \left| \begin{array}{cccc}
aA & dA & aD & dD \\
cA & bA & cD & bD \\
aC & dC & aB & dB \\
cC & bC & cB & bB
\end{array} \right| . \tag{34}
$$

This expression of elements of the group $SO(4.C)$ is analogue of isotropic form for Lorentz matrices in the Newman-Penrose formalism [11]. The above constrains (30) read

$$
ab - cd = +1\ , \qquad AB - CD = +1\ . \tag{35}
$$

Restriction to a real group $SO(4.R)$ is achieved by imposing four constrains:

$$
b = a^*\ , \qquad d = -c^*\ , \qquad B = A^*\ , \qquad D = -C^*\ . \tag{36}
$$

Let us specify the most interesting for physical applications the case of the real Lorentz group $O(3,1)$ with signature $(+---)$. To this end, let us turn again to (8):

$$R_\alpha(A_0, A_i) = A_0\ I + A_i \alpha^i = \left| \begin{array}{cccc} A_0 & A_1 & A_2 & A_3 \\ -A_1 & A_0 & -A_3 & A_2 \\ -A_2 & A_3 & A_0 & -A_1 \\ -A_3 & -A_2 & A_1 & A_0 \end{array} \right| ,$$

$$A''_0 = A'_0\ A_0 - A'_i\ A_i\ , \qquad A''_n = A'_0\ A_n + A'_n\ A_0 + (\epsilon_{ijn}\ A'_i\ A_j)\ ;$$

$$R_\beta(B_0, B_i) = B_0\ I + B_i \beta^i = \left| \begin{array}{cccc} B_0 & B_1 & B_2 & B_3 \\ -B_1 & B_0 & B_3 & -B_2 \\ -B_2 & -B_3 & B_0 & B_1 \\ -B_3 & B_2 & -B_1 & B_0 \end{array} \right| ,$$

$$B''_0 = B'_0\ B_0 - B'_i\ B_i\ , \qquad B''_n = B'_0\ B_n + B'_n\ B_0 - (\epsilon_{ijn}\ B'_i\ B_j)\ ;$$

they are valid in general for any complex parameters. On may obtain a subgroup by imposing restrictions

$$B_0 = A_0^*, \qquad B_k = -A_k^*\ ; \tag{37}$$

correspondingly we get

$$R(A, \bar{A}^*) = R_\alpha(A)\ R_\beta(\bar{A}^*) = (A_0\ I + A_i \alpha^i)\ (B_0\ I - A_j^* \beta^j) =$$
$$= A_0 A_0^* + A_0^* A_i\ \ \alpha^i - A_0 A_i^*\ \ \beta^i - A_i A_j^*\ \ \alpha^i \beta^j\ ; \tag{38}$$

$$A_0 A_0^* + A_0^* A_i\ \alpha^i - A_0 A_i^*\ \beta^i =$$

$$\left| \begin{array}{cccc} A_0 A_0^* & A_0^* A_1 - A_0 A_1^* & A_0^* A_2 - A_0 A_2^* & A_0^* A_3 - A_0 A_3^* \\ -A_0^* A_1 + A_0 A_1^* & A_0 A_0^* & -A_0^* A_3 - A_0 A_3^* & +A_0^* A_2 + A_0 A_2^* \\ -A_0^* A_2 + A_0 A_2^* & A_0^* A_3 + A_0 A_3^* & A_0 A_0^* & -A_0^* A_1 - A_0 A_1^* \\ -A_0^* A_3 + A_0 A_3^* & -A_0^* A_2 - A_0 A_2^* & +A_0^* A_1 + A_0 A_1^* & A_0 A_0^* \end{array} \right| ,$$

$$-A_i A_j^*\ \ \alpha^i \beta^j =$$

$$\left| \begin{array}{rrrr} A_j A_j^* & +A_2 A_3^* - A_3 A_2^* & -A_1 A_3^* + A_3 A_1^* & -A_1 A_2^* - A_2 A_1^* \\ +A_2 A_3^* - A_3 A_2^* & +A_1 A_1^* - A_2 A_2^* - A_3 A_3^* & A_1 A_2^* + A_2 A_1^* & +A_1 A_3^* + A_3 A_1^* \\ -A_1 A_3^* + A_3 A_1^* & +A_1 A_2^* + A_2 A_1^* & -A_1 A_1^* + A_2 A_2^* - A_3 A_3^* & +A_2 A_3^* + A_3 A_2^* \\ +A_1 A_2^* - A_2 A_1^* & +A_1 A_3^* + A_3 A_1^* & +A_2 A_3^* + A_3 A_2^* & -A_1 A_1^* - A_2 A_2^* + A_3 A_3^* \end{array} \right|$$

The matrices obtained $R(A, \bar{A}^*)$ determine transformations in 4-dimensional space with one real and three imaginary coordinates. Translating to real coordinates and real Lorentz matrices is achieved in the following way:

$$y'_a = R_{ab}\ y_b\ , \qquad \left| \begin{array}{c} y_0 \\ y_1 \\ y_2 \\ y_3 \end{array} \right| = \left| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \end{array} \right| \left| \begin{array}{c} x_0 \\ x_1 \\ x_2 \\ x_3 \end{array} \right| ,$$

$$Y = \Pi\ X\ , \qquad L(A, \bar{A}^*) = \Pi^{-1} R(A, \bar{A}^*) \Pi\ , \tag{39}$$

$$
\Pi^{-1}\,(A_0A_0^* + A_0^*A_i \;\; \alpha^i - A_0A_i^* \;\; \beta^i)\,\Pi =
$$

$$
\left|\begin{array}{cccc}
A_0A_0^* & i(A_0^*A_1 - A_0A_1^*) & i(A_0^*A_2 - A_0A_2^*) & i(A_0^*A_3 - A_0A_3^*) \\
i(A_0^*A_1 - A_0A_1^*) & A_0A_0^* & -A_0^*A_3 - A_0A_3^* & +A_0^*A_2 + A_0A_2^* \\
i(A_0^*A_2 - A_0A_2^*) & A_0^*A_3 + A_0A_3^* & A_0A_0^* & -A_0^*A_1 - A_0A_1^* \\
i(A_0^*A_3 - A_0A_3^*) & -A_0^*A_2 - A_0A_2^* & +A_0^*A_1 + A_0A_1^* & A_0A_0^*
\end{array}\right| ,
$$

$$
\Pi^{-1}\,(-A_iA_j^* \;\; \alpha^i\beta^j)\Pi =
$$

$$
\left|\begin{array}{rrrr}
A_jA_j^* & i(+A_2A_3^* - A_3A_2^*) & i(-A_1A_3^* + A_3A_1^*) & i(-A_1A_2^* - A_2A_1^*) \\
-i(+A_2A_3^* - A_3A_2^*) & A_1A_1^* - A_2A_2^* - A_3A_3^* & A_1A_2^* + A_2A_1^* & A_1A_3^* + A_3A_1^* \\
-i(-A_1A_3^* + A_3A_1^*) & A_1A_2^* + A_2A_1^* & A_2A_2^* - A_1A_1^* - A_3A_3^* & A_2A_3^* + A_3A_2^* \\
-i(+A_1A_2^* - A_2A_1^*) & +A_1A_3^* + A_3A_1^* & +A_2A_3^* + A_3A_2^* & A_3A_3^* - A_1A_1^* - A_2A_2^*
\end{array}\right| . \quad (40)
$$

Let us consider a particular case

$\underline{A_0 \neq 0\,, \qquad A_3 \neq 0}$

$$
L(A, \bar{A}^*) = \left|\begin{array}{cccc}
A_0A_0^* + A_3A_3^* & 0 & 0 & i(A_0^*A_3 - A_0A_3^*) \\
0 & A_0A_0^* - A_3A_3^* & -A_0^*A_3 - A_0A_3^* & 0 \\
0 & A_0^*A_3 + A_0A_3^* & A_0A_0^* - A_3A_3^* & 0 \\
i(A_0^*A_3 - A_0A_3^*) & 0 & 0 & A_0A_0^* + A_3A_3^*
\end{array}\right| . \qquad (41)
$$

At real parameters, it gives Euclidean rotations[2]:

$$
A_0^* = A_0 = D\cos\frac{\phi}{2}\,,\;\; A_3^* = A_3 = D\sin\frac{\phi}{2}\,,
$$

$$
L(A, \bar{A}^*) = D^2 \left|\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & \cos\phi & -\sin\phi & 0 \\
0 & \sin\phi & \cos\phi & 0 \\
0 & 0 & 0 & 1
\end{array}\right| ; \qquad (42)
$$

at complex parameters we have Lorentzian rotations:

$$
A_0 = A_0^* = D\,\mathrm{ch}\,\frac{\beta}{2}\,, \qquad A_3 = -A_3^* = iD\,\mathrm{sh}\,\frac{\beta}{2}\,,
$$

$$
L = \Pi^{-1}\, R\,\Pi = D^2 \left|\begin{array}{cccc}
\mathrm{ch}\,\beta & 0 & 0 & -\mathrm{sh}\,\beta \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-\mathrm{sh}\,\beta & 0 & 0 & \mathrm{ch}\,\beta
\end{array}\right| . \qquad (43)
$$

The quantity D determines determinant of L:

$$
\det L = D^8\,. \qquad (44)
$$

Let us consider another case:

[2]For needs in polarization optics it is important to have in hand extended Euclidean and Lorentzian rotations, including transformation with not-unit determinant – those transformations have a reasonable and separate physical interpretation.

$\underline{A_0 \neq 0 \, , \qquad A_1 \neq 0}$

$$L = \Pi^{-1}\; R\, \Pi = \left| \begin{array}{cccc} A_0 A_0^* + A_1 A_1^* & i(A_0^* A_1 - A_0 A_1^*) & 0 & 0 \\ i(A_0^* A_1 - A_0 A_1^*) & A_0 A_0^* + A_1 A_1^* & 0 & 0 \\ 0 & 0 & A_0 A_0^* - A_1 A_1^* & -A_0^* A_1 - A_0 A_1^* \\ 0 & 0 & +A_0^* A_1 + A_0 A_1^* & A_0 A_0^* - A_1 A_1^* \end{array} \right| , \tag{45}$$

from whence it follows:

$$A_0^* = A_0 = D \cos \frac{\phi}{2} \; , \; A_1^* = A_3 = D \sin \frac{\phi}{2} \; , \; L = D^2 \left| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\phi & -\sin\phi \\ 0 & 0 & \sin\phi & \cos\phi \end{array} \right| ; \tag{46}$$

$$A_0 = A_0^* = D \text{ ch } \frac{\beta}{2} \; , \; A_1 = -A_1^* = iD \text{ sh } \frac{\beta}{2} \; , \; L = D^2 \left| \begin{array}{cccc} \text{ch } \beta & -\text{ sh } \beta & 0 & 0 \\ -\text{ sh } \beta & \text{ch } \beta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right| . \tag{47}$$

Let us write down explicit form of two factors $R_\alpha(A_0, A_j)$ and $R_\beta(A_0^*, -A_j^*)$ after the similarity transformation:

$$\hat{A} = \Pi^{-1}\; R_\alpha(A_0, A_j)\Pi = \left| \begin{array}{cccc} A_0 & iA_1 & iA_2 & iA_3 \\ iA_1 & A_0 & -A_3 & A_2 \\ iA_2 & A_3 & A_0 & -A_1 \\ iA_3 & -A_2 & A_1 & A_0 \end{array} \right| ,$$

$$\hat{A}^* = \Pi^{-1}\; R_\beta(A_0^*, -A_j^*)\Pi = \left| \begin{array}{cccc} A_0^* & -iA_1^* & -iA_2^* & -iA_3^* \\ -iA_1^* & A_0^* & -A_3^* & A_2^* \\ -iA_2^* & A_3^* & A_0^* & -A_1^* \\ -iA_3^* & -A_2^* & A_1^* & A_0^* \end{array} \right| , \tag{48}$$

in other words, any Lorentz transformation $L(A, \hat{A}^*)$ is factorized into two commuting and conjugated matrices:

$$L = \hat{A}\; \hat{A}^* = \hat{A}^*\; \hat{A} \; . \tag{49}$$

In essence, just that factorized structure was used by Einstein and Mayer in its theory of semi-vectors [1, 2, 3]; later and more systematic construction of the Lorentz group theory in that base was given by Fedorov [4]. Also, it may be treated within the quaternion approach – for instance see [5], great list of references on quaternion in physics is given by Gsponer and Hurni [6, 7].

The most of facts exposed above in Section are not new ones, the role of this Section is to make present treatment self-consistent and intelligible and comprehensible for readers. In the next Sections we consider some applications of the above technique to description of the polarization of the light in Mueller and Jones formalisms.

3 Polarization of the light, Mueller 4-vector formalism

To elucidate how mathematical facts on rotation and Lorentz group exposed above may be applied to problems in polarization optics, and also what problems of this field wait to be solved, let us start with some basic definitions concerning the polarization of the light (at this we have used [15], though it might be another from many).

For a plane electromagnetic wave spreading along the axis z, in an arbitrary fixed point z we have (M, N are amplitudes of two electric components, Δ is a phase shift of two electric components)

$$E^1 = N \cos \omega t \;, \qquad E^2 = M \cos(\omega t + \Delta) \;, \qquad E^3 = 0 \;,$$
$$N \geq 0 \;, \qquad M \geq 0) \;, \qquad \Delta \in [-\pi, +\pi] \;; \tag{50}$$

four Stokes parameters $(S_a) = (I, S^1, S^2, S^3)$ are determined by definitions

$$I = < E_1^2 + E_2^2 > \;, \qquad S^3 = < E_1^2 - E_2^2 > \;,$$
$$S^1 = < 2E_1 E_2 \;\cos\Delta > \;, \qquad S^1 = < 2E_1 E_2 \;\sin\Delta > \;; \tag{51}$$

the symbol $< ... >$ stands for averaging in time. If the amplitudes N, M and the phase shift Δ do not depend on time in measuring process[3], the Stokes parameters equal to

$$S^0_{(pol)} = I_{(p)} = N^2 + M^2 \;, \qquad S^3_{(pol)} = N^2 - M^2 \;,$$
$$S^1_{(pol)} = 2NM \;\cos\Delta \;, \qquad S^2_{(pol)} = 2NM \;\sin\Delta \;, \tag{52}$$

and the identity holds

$$S_a S^a = (S^0_{(pol)})^2 - S^j_{(pol)} S^j_{(pol)} = I^2_{(pol)} - \mathbf{S}^2_{(pol)} = 0 \;; \tag{53}$$

that is $\mathbf{S} = I_{(pol)} \;\mathbf{n}$. In other words, for a completely polarized light, the Stoke 4-vector is isotropic.

For a natural (non-polarized) light, Stokes parameters are trivial:

$$S^a_{(non-pol)} = (I_{(non-pol)}, 0, 0, 0) \;.$$

When summing two non-coherent light waves, their Stokes parameters behave in accordance with the linear law $I_{(1)} + I_{(2)}$, $\mathbf{S}_{(1)} + \mathbf{S}_{(2)}$. In particular, a partly polarized light can be obtained as a linear sum of natural and completely polarized light:

$$S^a_{(non-pol)} = (I_{(non-pol)}, 0, 0, 0) \;, \qquad S^a_{(pol)} = (I_{(pol)}, I_{(pol)}\mathbf{n}) \;,$$

$$S^a = S^a_{(non-pol)} + S^a_{(pol)} = (\; I_{(non-pol)} + I_{(pol)} \;) \; \left(1, \frac{I_{(pol)}}{I_{(non-pol)} + I_{(pol)}} \;\mathbf{n}\right) \;,$$

with notation

$$I = I_{(non-pol)} + I_{(pol)} \;, \qquad p = \frac{I_{(pol)}}{I_{(non-pol)} + I_{(pol)}} \;, \tag{54}$$

[3] In this case we say about a completely polarized light; with that simplest case we will associate index (pol) below. There exist simple ways to obtain a completely polarized light experimentally, so the light with such properties is not something unattainable.

for the Stokes vector of the partly polarized light we have expressions

$$S^a = (I,\ I\ p\ \mathbf{n})\ , \qquad S_a S^a = I^2(1-p^2) \geq 0\ ; \tag{55}$$

where I is a general intensity, p is a degree of polarization which runs within [0, 1] interval: $0 \leq p \leq 1$, $\mathbf{n}$ stands for any 3-vector.

Behavior of Stokes 4-vectors for polarized and partly polarized light under acting optics devices may be considered as isomorphic to behavior of respectively isotropic and time-like vectors with respect to Lorentz group in Special Relativity. This simple observation leads to many consequences, some of them will be discussed below.

4 Polarized light and Jones formalism, restriction to symmetry $SU(2)$, and two sorts of non-relativistic Stokes 3-vectors

Now let us consider the Jones formalism and its connection with spinors for rotation and Lorentz groups. It is convenient to start with a relativistic 2-spinor ψ, representation of the special linear group $GL(2.C)$, covering for the Lorentz group:

$$\Psi = \left| \begin{array}{c} \psi^1 \\ \psi^2 \end{array} \right| , \qquad \Psi' = B(k)\Psi\ ,$$

$$B(k) = k_0 + k_j \sigma^j = \left| \begin{array}{cc} k_0 + k_3 & k_1 - ik_2 \\ k_1 + ik_2 & k_0 - k_3 \end{array} \right| = \left| \begin{array}{cc} a & d \\ c & b \end{array} \right| ,$$

$$\det = k_0^2 - \mathbf{k}^2 = ab - cd = 1\ , \qquad B(k) \in SL(2.C)\ . \tag{56}$$

From the spinor ψ one may construct a 2-rank spinor $\Psi \otimes \psi^*$, the 2×2 matrix, which in turn can be resolved in term of Pauli matrices. we will need two sets: $\sigma^a = (I, \sigma^j)$ and $\bar{\sigma}^a = (I, -\sigma^j)$. Let us decompose 2-rank spinor into the sum

$$\Psi \otimes \Psi^* = \frac{1}{2}\,(S_a\ \bar{\sigma}^a) = \frac{1}{2}\,(S_0 - S_j\ \sigma^j)\ . \tag{57}$$

The spinor nature of ψ will generate a definite transformation law for S_a:

$$(\Psi' \otimes {\Psi'}^*) = B(k)(\Psi' \otimes {\Psi'}^*)B^+(k) \qquad \Longrightarrow \qquad S'_a\ \bar{\sigma}^a = B(k) S_a\ \bar{\sigma}^a B^+(k)\ . \tag{58}$$

Now, one should use a well-known relation in the theory of the Lorentz group:

$$B(k)\bar{\sigma}^a B^+(k) = \bar{\sigma}^b L_b^{\ \ a} \tag{59}$$

where ${L_b}^a(x)$ is a 4×4 matrix, defined by

$${L_b}^a(x) = \frac{1}{2}\ \mathrm{sp}\ [\ \sigma_b\ B(k(x))\bar{\sigma}^a\ B(k^*(x))\ ] = {L_b}^a(k(x),\ k^*(x))\ , \tag{60}$$

With the use of the known formulas for traces of the Pauli matrices:

$$\frac{1}{2}\ \mathrm{sp}\ (\sigma_k \bar{\sigma}_l \sigma_a \bar{\sigma}_b) = g_{kl}g_{ab} - g_{ka}g_{lb} + g_{kb}g_{la} + i\epsilon_{klab}\ ,$$

for the matrix L we get

$$L_b{}^a(k,\ k^*) = \bar{\delta}_b^c\ [\ -\delta_c^a\ k^n\ k_n^*\ +\ k_c\ k^{a*}\ +\ k_c^*\ k^a\ +\ i\ \epsilon_c{}^{anm}\ k_n\ k_m^*\ ]\ ; \tag{61}$$

where

$$\bar{\delta}_b^c = \left\{ \begin{array}{l} 0, \quad c \neq b\ ; \\ +1, \quad c = b = 0\ ; \\ -1, \quad c = b = 1,\ 2,\ 3\ . \end{array} \right.$$

It should be noted that the matrix L used in Section transforms contra-variant vector, that is

$$L = L^a{}_b\ , \qquad U^a = L^a{}_b\, U^b, \qquad L^a{}_b = (L^{-1})_b{}^b$$

Substituting (59) into (58), one gets the transformation law for S_a:

$$S'_b = L_b{}^a\ S_a\ . \tag{62}$$

Thus, spinor transformation $B(k)$ for spinor ψ generates vector transformation $L_b{}^a(k, k^*)$. Different in sign spinor matrices, $\pm B$ lead to one the same matrix L. If we restrict ourselves to the case of SU(2) group, for matrix $L_b{}^a(k,\ k^*)$ we get:

$$k_0 = n_0\ , \qquad k_j = -in_j\ , \qquad n_0^2 + n_j n_j = +1\ , \qquad B(n) = n_0 - in_j\sigma_j\ ,$$

$$L(+n) = L(-n) = \left| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 - 2(n_2^2 + n_3^2) & -2n_0n_3 + 2n_1n_2 & 2n_0n_2 + 2n_1n_3 \\ 0 & 2n_0n_3 + 2n_1n_2 & 1 - 2(n_1^2 + n_3^2) & -2n_0n_1 + 2n_2n_3 \\ 0 & -2n_0n_2 + 2n_1n_3 & 2n_0n_1 + 2n_2n_3 & 1 - 2(n_1^2 + n_2^2) \end{array} \right| . \tag{63}$$

Let us introduce a polarization Jones spinor ψ:

$$\Psi = \left| \begin{array}{c} \Psi^1 \\ \psi^2 \end{array} \right| = \left| \begin{array}{c} N\ e^{i\alpha} \\ M\ e^{i\beta} \end{array} \right| , \qquad \psi \otimes \psi^* = \left| \begin{array}{cc} N^2 & NMe^{-i(\beta-\alpha)} \\ NMe^{+i(\beta-\alpha)} & M^2 \end{array} \right| =$$

$$= \frac{1}{2}(S_0 - S_j\ \sigma^j) = \frac{1}{2} \left| \begin{array}{cc} S_0 - S_3 & -S_1 + iS_2 \\ -S_1 - iS_2 & S_0 + S_3 \end{array} \right| = \frac{1}{2} \left| \begin{array}{cc} S^0 + S^3 & S^1 - iS^2 \\ S^1 + iS^2 & S^0 - S^3 \end{array} \right| , \tag{64}$$

that is

$$N^2 = \frac{1}{2}\ (S^0 + S^3)\ , \qquad M^2 = \frac{1}{2}\ (S^0 - S^3)\ ,$$
$$\frac{1}{2}\ (S^1 + iS^2) = NMe^{i(\beta-\alpha)}\ ,\ \frac{1}{2}\ (S^1 - iS^2) = NMe^{-i(\beta-\alpha)}\ .$$

From this it follows

$$S^1 = 2NM\cos(\beta - \alpha)\ , \qquad S^2 = 2NM\sin(\beta - \alpha)\ ,$$
$$S^3 = N^2 - M^2\ , \qquad S^0 = N^2 + M^2 = +\sqrt{S_1^2 + S_2^2 + S_3^2}\ . \tag{65}$$

They should be compared with eqs. (108)

$$S^0 = N^2 + M^2 = +\sqrt{S_1^2 + S_2^2 + S_3^2}\ ,$$
$$S^3 = N^2 - M^2\ , \qquad S^1 = 2NM\ \cos\Delta\ , \qquad S^2 = 2NM\ \sin\Delta\ ; \tag{66}$$

They coincide if $(\beta - \alpha) = \Delta$. Instead of α, β it is convenient introduce new variables:

$$\Delta = \beta - \alpha \; , \qquad \gamma = \beta + \alpha \; ,$$
$$\frac{1}{2}\,(\gamma + \Delta) = \beta \; , \qquad \frac{1}{2}\,(\gamma - \Delta) = \alpha \; ;$$

correspondingly the spinor Ψ will look (139)

$$\Psi = e^{i\gamma/2} \left| \begin{array}{c} N\, e^{-i\Delta/2} \\ M\, e^{+i\Delta/2} \end{array} \right| = e^{i\gamma/2} \left| \begin{array}{c} \sqrt{(S+S^3)/2}\;\; e^{-i\Delta/2} \\ \sqrt{(S-S^3)/2}\;\; e^{+i\Delta/2} \end{array} \right| . \tag{67}$$

remembering that Jones complex 2-vector, it is just another representation of the electric field in the plane electromagnetic wave (see (106)):

$$E^1 = N\cos\omega t \; , \qquad E^2 = M\cos(\omega t + \Delta) \; , \qquad E^3 = 0 \; .$$

Let us write down the inverse relations to (65) – (66), they are

$$2N^2 = S + S^3 \; , \qquad 2M^2 = S - S^3 \; , \qquad \Delta = \text{arctg}\,\frac{S^2}{S^1} \; ; \tag{68}$$

these correlate with the known relations defining parabolic coordinates

$$\xi = r + z \; , \qquad \eta = r - z \; , \qquad \phi = \text{arctg}\,\frac{y}{x} \; . \tag{69}$$

Evidently, we have isomorphism between parameters of the Jones spinor (N, M, Δ) and parabolic coordinate (ξ, η, δ) in effective space of Stokes 3-vector (S^1, S^2, S^3):

$$\xi = 2N^2 \; , \qquad \eta = 2M^2 \; , \qquad \phi = \Delta \; .$$
$$x = S^1 \; , \qquad y = S^2 \; , \qquad z = S^3 \; . \tag{70}$$

One could pose the task to find space spinor Ψ_{space} related to space coordinates (x, y, z) on the base of relationship (see Cartan [14])

$$\Psi_{space} = \left| \begin{array}{c} N\, e^{i\alpha} \\ M\, e^{i\beta} \end{array} \right| , \qquad \psi \otimes \psi^* = \frac{1}{2} \left| \begin{array}{cc} r+z & x-iy \\ x+iy & r-z \end{array} \right| , \tag{71}$$

one would produce the formulas (for more detail see [12], [13]) like (67):

$$\Psi_{space} = e^{-\gamma/2} \left| \begin{array}{c} \sqrt{r+z}\; e^{-i\phi/2} \\ \sqrt{r-z}\; e^{+i\phi/2} \end{array} \right| = e^{-\gamma/2} \left| \begin{array}{c} \sqrt{\xi}\; e^{-i\phi/2} \\ \sqrt{\eta}\; e^{+i\phi/2} \end{array} \right| , \quad e^{i\phi} = \frac{x+iy}{\sqrt{x^2+y^2}} \; . \tag{72}$$

Spinor Ψ (or Ψ_{space}) has evident peculiarity: at the whole axis $S_1 = S_2 = 0$ (or at $x = y = 0$) its defining relation contains ambiguity $(0 + i0)/0$ (and expressions for ξ will contain a mute angle variable $\Gamma :\; \phi \rightarrow \Gamma$)

$$(0, 0, S_3 > 0) \;\; : \qquad \Psi_0^+ = \left| \begin{array}{c} \sqrt{+2S_3}\; e^{-i\Delta/2} \\ 0 \end{array} \right| ,$$
$$(0, 0, S_3 < 0) \;\; : \qquad \Psi_0^- = \left| \begin{array}{c} 0 \\ \sqrt{-2S_3}\; e^{+i\Delta/2} \end{array} \right| ,$$
$$e^{i\Gamma} = \lim_{S_1, S_2 \rightarrow 0} \frac{S_1 + iS_2}{\sqrt{S_1^2 + S_2^2}} \; . \tag{73}$$

It should be mention that polarization singularities, attracting attention in the literature [116], should be associated with peculiarities $(0+i0)/0$ in (73); in other words, it is peculiarity in parameterizing space spinor Ψ_{space} by parabolic coordinates (ξ,η,ϕ).

Also one can take special attention to the factors $e^{+i\phi/2}$ and $e^{-i\phi/2}$ in expression for Jones spinor, which leads to $(\pm)$-ambiguity at the values $\Phi=0$ and $\Phi=+2\pi$ or $\Delta=0$ and $\Delta=+2\pi$. However these two values correspond physically to one the same direction in geometrical space or to one the same Stokes 3-vector. It is an old problem with spinors applied to description of 3-vectors, and it can be overcome in the frame ideas on spinor space structure: [11]; also see in [12], [13] and references therein. Does the spinor group topology is relevant indeed to Jones complex formalism in optics or not – the issue remains open, for theory and experiments.

Also, we might take special attention to the fact, that usually the space vector (x,y,z) is not taken to be a pseudovector, but constructing (x,y,z) in spinor approach according to (71) leads just to a such pseudo-vector.

Let us recall Cartan classification for non-relativistic spinors [14] of 2-spinors with respect to spinor P-reflection: namely, the simplest irreducible representations of the unitary extended group

$$\tilde{SU}(2)=\Big\{\; g\in SU(2)\oplus J=\left|\begin{array}{cc} i & 0\\ 0 & i\end{array}\right|,\;\; \det\; g=+1,\; \det\; J=-1\;\Big\}$$

are 2-component spinors of two types T_1,T_2

$$T_1(g)=g\;,\;\; T_1(J)=+J\;; \qquad\qquad T_2(g)=g\;,\;\; T_2(J)=-J\;.$$

There are two ways to construct 3-vector (complex-valued in general) in terms of 2-spinors

$$\begin{array}{ll} 1. & (\Psi_{space}\otimes\Psi^{*}_{space})=r\;+\;x_j\;\sigma^j\;\;,\;\; r=+\sqrt{x_j\;x_j}\;, \qquad x_j-\text{pseudovector}\;;\\ 2. & (\Psi'_{space}\otimes\Psi'_{space})=\;(y_j\;+\;i\;x_j)\;\sigma^j\;\sigma^2\;, \qquad y_j,x_j-\text{vectors}\;. \end{array} \tag{74}$$

Evidently, variant 1 provides us with possibility to build a spinor model for pseudo vector 3-space, whereas variant 2 leads to a spinor model of properly vector 3-space. In other words, according to different ways of taking the square root of three real numbers – components of a 3-vector x_i – one will arrive at two different spatial spinors:

$$\Psi_{space}\;\Longleftrightarrow\; x_j\;, \qquad \Psi'_{space}\;\Longleftrightarrow\; x_j\;.$$

These spinors, Ψ_{space} and Ψ'_{space} respectively, turned out to be different functions of Cartesian coordinates. In particular, the second spinor model corresponding to a vector space, variant 2. in (74), is described by two spinors $\Psi'_{space}(\vec{x})$, each covering a vector half-space (for more detail see [12], [13]):

$$b_3>0\;, \qquad H^{+}_{space}=\left|\begin{array}{c}\sqrt{r-(x^2+y^2)^{1/2}}\;e^{-i\phi/2}\\ \sqrt{r+(x^2+y^2)^{1/2}}\;e^{+i\phi/2}\end{array}\right|, \qquad e^{i\phi}=\frac{x+iy}{\sqrt{x^2+y^2}}\;.$$

$$b_3<0\;, \qquad H^{-}_{space}=i\left|\begin{array}{c}\sqrt{r-(x^2+y^2)^{1/2}}\;e^{-i\sigma/2}\\ \sqrt{r+(x^2+y^2)^{1/2}}\;e^{+i\sigma/2}\end{array}\right|, \qquad e^{i\sigma/2}=-i\sqrt{\frac{x+iy}{\sqrt{x^2+y^2}}}\;. \tag{75}$$

In the context of polarization optics instead of (75) we would have

$$S^3 > 0 \; , \qquad \Psi'_+ = \left| \begin{array}{c} N'\; e^{-i\Delta/2} \\ M'\; e^{+i\Delta/2} \end{array} \right| = \left| \begin{array}{c} \sqrt{S - (S_1^2 + S_2^2)^{1/2}}\; e^{-i\Delta/2} \\ \sqrt{S + (S_1^2 + S_2^2)^{1/2}}\; e^{+i\Delta/2} \end{array} \right| \; , \;\; e^{i\Delta} = \frac{S^1 + iS^2}{\sqrt{S_1^2 + S_2^2}} \; .$$

$$S^3 < 0 \; , \qquad \Psi'_- = \left| \begin{array}{c} N'\; e^{-i\Delta/2} \\ M'\; e^{+i\Delta/2} \end{array} \right| = i \left| \begin{array}{c} \sqrt{S - (S_1^2 + S_2^2)^{1/2}}\; e^{-i\sigma/2} \\ \sqrt{S + (S_1^2 + S_2^2)^{1/2}}\; e^{+i\sigma/2} \end{array} \right| \; , \;\; e^{i\sigma/2} = -i \sqrt{\frac{S^1 + iS^2}{\sqrt{S_1^2 + S_2^2}}} \; . \tag{76}$$

Here again we have singularity in parametrization on the whole axis $b_1 = 0, b_2 = 0$ or at $S^1 = 0,\; S^2 = 0$.

Two models of spinors spaces with respect to P-orientation are grounded on different mappings $\Psi(\vec{x})$ and $\Psi'_{space}(\vec{x})$ defined over the same extended domain $\tilde{G}(y_i)$. The natural question is: how are these two maps connected to each others. An answer can be found on comparing the formulas

$$\Psi_{space} = \left| \begin{array}{c} \sqrt{\xi}\; e^{-i\phi/2} \\ \sqrt{\eta}\; e^{+i\phi/2} \end{array} \right| \; , \qquad \Psi'_{space} = \frac{1}{\sqrt{2}} \left| \begin{array}{c} (\sqrt{\xi} - \sqrt{\eta})\; e^{-i\phi/2} \\ (\sqrt{\xi} + \sqrt{\eta})\; e^{+i\phi/2} \end{array} \right| \; . \tag{77}$$

From (77) we immediately arrive at

$$\Psi'_{space} = \frac{1}{\sqrt{2}} (\Psi_{space} - i\; \sigma^2 \xi^*_{space}) \; ,$$
$$\Psi_{space} = \frac{1}{\sqrt{2}}\; (\Psi'_{space} - i\; \sigma^2 \Psi'^{*}_{space}) \; . \tag{78}$$

Finally, let us write down the formulas for Stoke 3-vector in both cases:

traditional model $\Psi(\vec{S})$

$$S^1 = \sqrt{\frac{NM}{2}}\; \cos\Delta \; , \qquad S^2 = \sqrt{\frac{NM}{2}}\; \sin\Delta \; , \qquad S^3 = N^2 - M^2 \; ;$$

alternative model $\Psi'(\vec{S})$

$$S^1 = \sqrt{2 \mid M'^2 - N'^2 \mid}\; \cos\Delta \; ,$$
$$S^2 = \sqrt{2 \mid M'^2 - N'^2 \mid}\; \sin\Delta \; , \qquad S^3 = \pm \sqrt{N'M'} \; . \tag{79}$$

5 Factorizations for 3-rotations and polarization of the light

In this Section we will consider several problems naturally arising within the task of finding different ways to parameterize the unitary SU(2) and rotation SO(3) groups by three angle variables. There exist two different classes of those. The first based on 2-element factorizations:

$$(U_2 U_3 U'_2), \qquad (U_3 U_2 U'_3), \qquad (U_3 U_1 U'_3),$$
$$(U_1 U_3 U'_1), \qquad (U_1 U_2 U'_1), \qquad (U_2 U_1 U'_2) \; ,$$

provides all possible ways to define Euler's angles. The second based on 3-element factorizations:

$$
\begin{array}{lll}
(U_1U_2U_3), & (U_1U_3U_2'), & (U_2U_3U_1), \\
(U_2U_1U_3), & (U_3U_1U_2), & (U_3U_2U_1) \; .
\end{array}
$$

In the literature, this second possibility is used rarely, and as a rule is only pointed out as existing one.

This rather abstract group theoretical problems have a definite sense in the context of the light polarization formalism of Stokes-Mueller vectors and Jones spinors. Because relations obtained give a base to resolve arbitrary pure polarization rotators into all possible sets of elementary rotators of two or tree types.

2-element factorization, one special case

In this Section we consider the 2-element factorizations:

$$
\begin{aligned}
U &= U_1U_2U_1' \\
&= (x_0 + ix_1\sigma_1)(y_0 + iy_2\sigma_2)(x_0' + ix_1'\sigma_1) \\
&= y_0\;(x_0x_0' - x_1x_1') \\
&\quad + i\sigma_1\; y_0\;(x_1x_0' + x_0x_1') \\
&\quad + i\sigma_2\; y_2\;(x_0x_0' + x_1x_1') \\
&\quad + i\sigma_3\; y_2\;(-x_1x_0' + x_0x_1') \\
&= n_0 + in_1\,\sigma_1 + in_2\,\sigma_2 + in_3\,\sigma_3 \; ,
\end{aligned}
\tag{80}
$$

that is

$$
\begin{array}{ll}
n_0 = y_0\;(x_0x_0' - x_1x_1') \; , & n_1 = y_0\;(x_1x_0' + x_0x_1') \; , \\
n_2 = y_2\;(x_0x_0' + x_1x_1') \; , & n_3 = y_2\;(-x_1x_0' + x_0x_1') \; .
\end{array}
\tag{81}
$$

We should resolve eqs. (81) under the variables $x_0, x_1; x_0', x_1'; y_0, y_2$. For y_0, y_2 we have

$$
\begin{aligned}
y_0 &= \frac{n_0}{(x_0x_0' - x_1x_1')} = \frac{n_1}{(x_1x_0' + x_0x_1')} \; , \\
y_2 &= \frac{n_2}{(x_0x_0' + x_1x_1')} = \frac{n_3}{(-x_1x_0' + x_0x_1')} \; .
\end{aligned}
\tag{82}
$$

In eqs. (81), one can exclude the variables y_0 and y_2:

$$
\frac{n_0}{n_1} = \frac{(x_0x_0' - x_1x_1')}{(x_1x_0' + x_0x_1')} \; , \qquad \frac{n_2}{n_3} = \frac{(x_0x_0' + x_1x_1')}{(-x_1x_0' + x_0x_1')}
$$

or

$$
\begin{aligned}
n_0\;(x_1x_0' + x_0x_1') &= n_1\;(x_0x_0' - x_1x_1') \; , \\
n_2\;(-x_1x_0' + x_0x_1') &= n_3\;(x_0x_0' + x_1x_1') \; .
\end{aligned}
\tag{83}
$$

Eqs. (83) can be resolved as a linear system under the variables x_0, x_1

$$
I \qquad \left\{ \begin{array}{l}
x_0\;(n_0x_1' - n_1x_0') + x_1\;(n_0x_0' + n_1x_1') = 0 \; , \\
x_0\;(n_2x_1' - n_3x_0') + x_1\;(-n_2x_0' - n_3x_1') = 0 \; ,
\end{array} \right.
\tag{84}
$$

or under the variables x'_0, x'_1:

$$
II \qquad \left\{ \begin{array}{l} x'_0\ (n_0 x_1 - n_1 x_0) + x'_1\ (n_0 x_0 + n_1 x_1) = 0\ , \\ x'_0\ (-n_2 x_1 - n_3 x_0) + x'_1\ (n_2 x_0 - n_3 x_1) = 0\ . \end{array} \right. \tag{85}
$$

First, let us study eqs. (84):

$$
I \qquad \det \left| \begin{array}{cc} (n_0 x'_1 - n_1 x'_0) & (n_0 x'_0 + n_1 x'_1) \\ (n_2 x'_1 - n_3 x'_0) & (-n_2 x'_0 - n_3 x'_1) \end{array} \right| = 0\ , \qquad \Longrightarrow
$$

$$
(n_1 n_3 - n_0 n_2)\ 2 x'_0 x_1 + (n_0 n_3 + n_1 n_2)(x_0^{'2} - x_1^{'2}) = 0\ . \tag{86}
$$

With the use of angle parametrization for x'_0, x'_1: $x'_0 = \cos \frac{a'}{2}$, $x'_1 = \sin \frac{a'}{2}$, eq. (86) takes the form

$$
(n_1 n_3 - n_0 n_2)\ \sin a' + (n_0 n_3 + n_1 n_2)\ \cos a' = 0\ ,
$$

that is

$$
\text{tg}\ a' = \frac{n_0 n_3 + n_1 n_2}{n_0 n_2 - n_1 n_3}\ , \tag{87}
$$

and further

$$
\begin{array}{c} \cos a' = \sqrt{\dfrac{1}{1 + \text{tg}^2\ a'}} = \pm \dfrac{(n_0 n_2 - n_1 n_3)}{\sqrt{n_0^2 + n_1^2}\sqrt{n_2^2 + n_3^2}}\ , \\ \sin a' = \cos a'\ \text{tg}\ a' = \pm \dfrac{(n_0 n_3 + n_1 n_2)}{\sqrt{n_0^2 + n_1^2}\sqrt{n_2^2 + n_3^2}}\ . \end{array} \tag{88}
$$

Thus

$$
x'_0 = \mu' \cos \frac{a'}{2} = \sqrt{\frac{1 + \cos a'}{2}}\ , \quad \mu' = \pm 1\ , \qquad x'_1 = \delta \sin \frac{a'}{2} = \sqrt{\frac{1 - \cos a'}{2}}\ , \quad \delta' = \pm 1\ ,
$$

or

$$
x'_0 = \mu'\ \sqrt{\frac{1}{2} \pm \frac{n_0 n_2 - n_1 n_3}{2\sqrt{n_0^2 + n_1^2}\sqrt{n_2^2 + n_3^2}}}\ , \qquad x'_1 = \delta'\ \sqrt{\frac{1}{2} \mp \frac{n_0 n_2 - n_1 n_3}{2\sqrt{n_0^2 + n_1^2}\sqrt{n_2^2 + n_3^2}}}\ . \tag{89}
$$

Turning to eqs. (84), we get two (equivalent) solutions:

$$
x_0 = \pm \frac{(n_0 x'_0 + n_1 x'_1)}{\sqrt{(n_0 x'_1 - n_1 x'_0)^2 + (n_0 x'_0 + n_1 x'_1)^2}}\ , \qquad x_1 = \pm \frac{(n_1 x'_0 - n_0 x'_1)}{\sqrt{(n_0 x'_1 - n_1 x'_0)^2 + (n_0 x'_0 + n_1 x'_1)^2}}\ ; \tag{90}
$$

$$
x_0 = \pm \frac{(n_2 x'_0 + n_3 x'_1)}{\sqrt{(n_2 x'_1 - n_3 x'_0)^2 + (-n_2 x'_0 - n_3 x'_1)^2}}\ , \qquad x_1 = \pm \frac{(n_2 x'_1 - n_3 x'_0)}{\sqrt{(n_2 x'_1 - n_3 x'_0)^2 + (n_2 x'_0 + n_3 x'_1)^2}}\ ; \tag{91}
$$

Now let us make the same with the system (85), which formally differs from (84) by evident changes

$$
n_2 \longrightarrow -n_2\ , \qquad x_a \longrightarrow x'_a\ ;
$$

from vanishing the determinant

$$II \qquad \det \begin{vmatrix} (n_0x_1 - n_1x_0) & (n_0x_0 + n_1x_1) \\ (-n_2x_1 - n_3x_0) & (n_2x_0 - n_3x_1) \end{vmatrix} = 0, \tag{92}$$

we get

$$\begin{gathered} \text{tg } a = \frac{-n_0n_3 + n_1n_2}{n_0n_2 + n_1n_3} \, , \\ \cos a = \pm\sqrt{\frac{1}{1 + \text{tg}^2\, a}} = \pm\frac{(n_0n_2 + n_1n_3)}{\sqrt{n_0^2 + n_1^2}\sqrt{n_2^2 + n_3^2}} \, , \\ \sin a = \cos a \text{ tg } a = \pm\frac{(-n_0n_3 + n_1n_2)}{\sqrt{n_0^2 + n_1^2}\sqrt{n_2^2 + n_3^2}} \, ; \end{gathered} \tag{93}$$

also

$$x_0 = \cos\frac{a}{2} = \mu\sqrt{\frac{1 + \cos a}{2}} \, , \ \ \mu = \pm 1 \, , \qquad x_1 = \sin\frac{a}{2} = \delta\sqrt{\frac{1 - \cos a}{2}} \, , \ \ \delta = \pm 1 \, , \tag{94}$$

$$x_0 = \mu \, \sqrt{\frac{1}{2} \pm \frac{n_0n_2 + n_1n_3}{2\sqrt{n_0^2 + n_1^2}\sqrt{n_2^2 + n_3^2}}} \, , \qquad x_1 = \delta \, \sqrt{\frac{1}{2} \mp \frac{n_0n_2 + n_1n_3}{2\sqrt{n_0^2 + n_1^2}\sqrt{n_2^2 + n_3^2}}} \, ,. \tag{95}$$

Turning to the system (85), we produce two (equivalent) solutions:

$$x_0' = \pm\frac{(n_0x_0 + n_1x_1)}{\sqrt{(n_0x_1 - n_1x_0)^2 + (n_0x_0 + n_1x_1)^2}} \, , \qquad x_1' = \mp\frac{(n_0x_1 - n_1x_0)}{\sqrt{(n_0x_1 - n_1x_0)^2 + (n_0x_0 + n_1x_1)^2}} \, , \tag{96}$$

$$x_0' = \pm\frac{(n_2x_0 - n_3x_1)}{\sqrt{(-n_2x_1 - n_3x_0)^2 + (n_2x_0 - n_3x_1)^2}} \, , \qquad x_1' = \mp\frac{(n_0x_1 - n_1x_0)}{\sqrt{(-n_2x_1 - n_3x_0)^2 + (n_2x_0 - n_3x_1)^2}} \, . \tag{97}$$

Evidently, the systems, I and II, are equivalent, so they must provides us with the same solutions – collect results together:

I

$$\cos a' = \pm\frac{(n_0n_2 - n_1n_3)}{\sqrt{n_0^2 + n_1^2}\sqrt{n_2^2 + n_3^2}} \, , \ \sin a' = \pm\frac{(n_0n_3 + n_1n_2)}{\sqrt{n_0^2 + n_1^2}\sqrt{n_2^2 + n_3^2}} \, , \ \text{tg } a' = \frac{n_0n_3 + n_1n_2}{n_0n_2 - n_1n_3} \, ;$$

$$x_0' = \mu' \, \sqrt{\frac{1}{2} \pm \frac{n_0n_2 - n_1n_3}{2\sqrt{n_0^2 + n_1^2}\sqrt{n_2^2 + n_3^2}}} \, , \qquad x_1' = \delta' \, \sqrt{\frac{1}{2} \mp \frac{n_0n_2 - n_1n_3}{2\sqrt{n_0^2 + n_1^2}\sqrt{n_2^2 + n_3^2}}} \, ;$$

$$\begin{gathered} x_0 = \pm\frac{(n_0x_0' + n_1x_1')}{\sqrt{n_0^2 + n_1^2}} \, , \qquad x_1 = \pm\frac{(n_1x_0' - n_0x_1')}{\sqrt{n_0^2 + n_1^2}} \, ; \\ x_0 = \pm\frac{(n_2x_0' + n_3x_1')}{\sqrt{n_2^2 + n_3^2}} \, , \qquad x_1 = \pm\frac{(n_2x_1' - n_3x_0')}{\sqrt{n_2^2 + n_3^2}} \, ; \end{gathered} \tag{98}$$

II

$$\cos a = \pm \frac{(n_0 n_2 + n_1 n_3)}{\sqrt{n_0^2+n_1^2}\sqrt{n_2^2+n_3^2}} \;, \; \sin a = \pm \frac{(-n_0 n_3 + n_1 n_2)}{\sqrt{n_0^2+n_1^2}\sqrt{n_2^2+n_3^2}} \;, \text{tg } a = \frac{-n_0 n_3 + n_1 n_2}{n_0 n_2 + n_1 n_3} \;,$$

$$x_0 = \mu \sqrt{\frac{1}{2} \pm \frac{n_0 n_2 + n_1 n_3}{2\sqrt{n_0^2+n_1^2}\sqrt{n_2^2+n_3^2}}} \;, \qquad x_1 = \delta \sqrt{\frac{1}{2} \mp \frac{n_0 n_2 + n_1 n_3}{2\sqrt{n_0^2+n_1^2}\sqrt{n_2^2+n_3^2}}} \;;$$

$$x_0' = \pm \frac{(n_0 x_0 + n_1 x_1)}{\sqrt{n_0^2+n_1^2}} \;, \;\; x_1' = \mp \frac{(n_0 x_1 - n_1 x_0)}{\sqrt{n_0^2+n_1^2}} \;;$$

$$x_0' = \pm \frac{(n_2 x_0 - n_3 x_1)}{\sqrt{n_2^2+n_3^2}} \;, \qquad x_1' = \mp \frac{(n_0 x_1 - n_1 x_0)}{\sqrt{n_2^2+n_3^2}} \;. \tag{99}$$

For a time, giving up more complex forms, let us use the most simple ones (one may omit $\pm$)

for a, a':

$$\cos a' = \frac{(n_0 n_2 - n_1 n_3)}{\sqrt{n_0^2+n_1^2}\sqrt{n_2^2+n_3^2}} \;, \quad \sin a' = \frac{(n_0 n_3 + n_1 n_2)}{\sqrt{n_0^2+n_1^2}\sqrt{n_2^2+n_3^2}} \;;$$

$$\cos a = \frac{(n_0 n_2 + n_1 n_3)}{\sqrt{n_0^2+n_1^2}\sqrt{n_2^2+n_3^2}} \;, \quad \sin a = \frac{(-n_0 n_3 + n_1 n_2)}{\sqrt{n_0^2+n_1^2}\sqrt{n_2^2+n_3^2}} \;; \tag{100}$$

and for $\frac{a}{2}, \frac{a'}{2}$:

$$x_0' = \cos\frac{a'}{2} = \mu' \sqrt{\frac{1}{2} + \frac{n_0 n_2 - n_1 n_3}{2\sqrt{n_0^2+n_1^2}\sqrt{n_2^2+n_3^2}}} \;, \quad x_1' = \sin\frac{a'}{2} = \delta' \sqrt{\frac{1}{2} - \frac{n_0 n_2 - n_1 n_3}{2\sqrt{n_0^2+n_1^2}\sqrt{n_2^2+n_3^2}}} \;;$$

$$x_0 = \cos\frac{a}{2} = \mu \sqrt{\frac{1}{2} + \frac{n_0 n_2 + n_1 n_3}{2\sqrt{n_0^2+n_1^2}\sqrt{n_2^2+n_3^2}}} \;, \qquad x_1 = \sin\frac{a}{2} = \delta \sqrt{\frac{1}{2} - \frac{n_0 n_2 + n_1 n_3}{2\sqrt{n_0^2+n_1^2}\sqrt{n_2^2+n_3^2}}} \;; \tag{101}$$

Turning to eqs. (82), we should calculate expressions for

$$y_0 = \frac{n_0}{(x_0 x_0' - x_1 x_1')} = \frac{n_1}{(x_1 x_0' + x_0 x_1')} \;,$$
$$y_2 = \frac{n_2}{(x_0 x_0' + x_1 x_1')} = \frac{n_3}{(-x_1 x_0' + x_0 x_1')} \;. \tag{102}$$

Let us introduce angle variable $y_0 = \cos\frac{b}{2}$, $y_2 = \sin\frac{b}{2}$ and calculate $\sin b$ and $\cos b$; for definiteness let us use second expressions in (102):

$$\sin b = 2 y_0 y_2 = 2 \; \frac{n_1 n_3}{x_1^2(-1 + x_1'^2) + x_0^2 x_1'^2} = 2 \; \frac{n_1 n_3}{x_1'^2 - x_1^2} \;;$$

allowing for identity

$$x_1'^2 - x_1^2 = \left(\frac{1}{2} - \frac{n_0 n_2 - n_1 n_3}{2\sqrt{n_0^2+n_1^2}\sqrt{n_2^2+n_3^2}}\right) - \left(\frac{1}{2} - \frac{n_0 n_2 + n_1 n_3}{2\sqrt{n_0^2+n_1^2}\sqrt{n_2^2+n_3^2}}\right) = \frac{n_1 n_3}{\sqrt{n_0^2+n_1^2}\sqrt{n_2^2+n_3^2}}$$

we arrive at

$$\sin b = 2\,\sqrt{n_0^2+n_1^2}\,\sqrt{n_2^2+n_3^2}\,. \tag{103}$$

It is readily verified that the result will be the same if one uses first expressions in (102). Let us calculate $\cos b$:

$$\cos^2 b = 1-\sin^2 b = (n_0^2+n_1^2)+(n_2^2+n_3^2)-4(n_0^2+n_1^2)\,(n_2^2+n_3^2) = (n_0^2+n_1^2-n_2^2-n_3^2)^2\,,$$

so that

$$\cos b = (n_0^2+n_1^2-n_2^2-n_3^2)\,. \tag{104}$$

Now let us calculate y_0 and y_2 from (102). Allowing for

$$(x_0x_0' \pm x_1x_1') = \frac{1}{2\sqrt{n_0^2+n_1^2}\,\sqrt{n_2^2+n_3^2}}\times$$
$$\times[\,(\sqrt{n_0^2+n_1^2}\sqrt{n_2^2+n_3^2}+(n_0n_2+n_1n_3))^{1/2}\,(\sqrt{n_0^2+n_1^2}\sqrt{n_2^2+n_3^2}+(n_0n_2-n_1n_3))^{1/2}\pm$$
$$\pm\,(\sqrt{n_0^2+n_1^2}\sqrt{n_2^2+n_3^2}-(n_0n_2+n_1n_3))^{1/2}\,(\sqrt{n_0^2+n_1^2}\sqrt{n_2^2+n_3^2}-(n_0n_2-n_1n_3))^{1/2}\,]=$$
$$=\frac{1}{2\sqrt{n_0^2+n_1^2}\,\sqrt{n_2^2+n_3^2}}\,[\,(2n_0^2n_2^2+n_0^2n_3^2+n_1^2n_2^2+2n_0n_2\sqrt{n_0^2+n_1^2}\sqrt{n_2^2+n_3^2})^{1/2}\pm$$
$$\pm(2n_0^2n_2^2+n_0^2n_3^2+n_1^2n_2^2-2n_0n_2\sqrt{n_0^2+n_1^2}\sqrt{n_2^2+n_3^2})^{1/2}\,]\,,$$

so that

$$(x_0x_0'+x_1x_1') = \frac{(n_0\,\sqrt{n_2^2+n_3^2}+n_2\,\sqrt{n_0^2+n_1^2}\,)-(n_0\,\sqrt{n_2^2+n_3^2}-n_2\,\sqrt{n_0^2+n_1^2}\,)}{2\sqrt{n_0^2+n_1^2}\,\sqrt{n_2^2+n_3^2}} = \frac{n_2}{\sqrt{n_2^2+n_3^2}}\,,$$
$$(x_0x_0'-x_1x_1') = \frac{(n_0\,\sqrt{n_2^2+n_3^2}+n_2\,\sqrt{n_0^2+n_1^2}\,)+(n_0\,\sqrt{n_2^2+n_3^2}-n_2\,\sqrt{n_0^2+n_1^2}\,)}{2\sqrt{n_0^2+n_1^2}\,\sqrt{n_2^2+n_3^2}} = \frac{n_0}{\sqrt{n_0^2+n_1^2}}\,,$$

thus

$$y_0 = \frac{n_0}{(x_0x_0'-x_1x_1')} = \sqrt{n_2^2+n_3^2}\,, \qquad y_2 = \frac{n_2}{(x_0x_0'+x_1x_1')} = \sqrt{n_0^2+n_1^2}\,. \tag{105}$$

All six types of 2-element factorization

There exist six types of 2-element factorization:

$$\begin{array}{llll} (1) & U_2\,U_3\,U_2'\,, & (1)' & U_3\,U_2\,U_3'\,; \\ (2) & U_3\,U_1\,U_3'\,, & (2)' & U_1\,U_3\,U_1'\,; \\ (3) & \underline{U_1\,U_2\,U_1'}\,, & (3)' & U_2\,U_1\,U_2'\,. \end{array} \tag{106}$$

in the previous Section we considered one (underlines) factorization $\underline{U_1\,U_2\,U_1'}$. We are to extend the above analysis to all six variants:

$$U = n_0 + in_1\,\sigma_1 + in_2\,\sigma_2 + in_3\,\sigma_3\,,$$

$$\begin{array}{ll}
(1) & U = U_2U_3U_2' = (y_0 + iy_2\sigma_2)(z_0 + iz_3\sigma_3)(y_0' + iy_2'\sigma_2)\;, \\
& n_0 = z_0\;(y_0y_0' - y_2y_2')\;, \qquad n_3 = z_3\;(y_0y_0' + y_2y_2')\;, \\
& n_2 = z_0\;(y_0y_2' + y_2y_0')\;, \qquad n_1 = z_3\;(y_0y_2' - y_2y_0')\;. \\
(1)' & U = U_3U_2U_3' = (z_0 + iz_3\sigma_3)(y_0 + iy_2\sigma_2)(z_0' + iz_3'\sigma_3)\;, \\
& n_0 = y_0\;(z_0z_0' - z_3z_3')\;, \qquad n_2 = y_2\;(z_0z_0' + z_3z_3')\;, \\
& n_3 = y_0\;(z_0z_3' + z_3z_0')\;, \qquad -n_1 = y_2\;(z_0z_3' - z_3z_0')\;;
\end{array} \tag{107}$$

$$\begin{array}{ll}
(2) & U = U_3U_1U_3' = (z_0 + iz_3\sigma_3)(x_0 + ix_1\sigma_1)(z_0' + iz_3'\sigma_3)\;, \\
& n_0 = x_0\;(z_0z_0' - z_3z_3')\;, \qquad n_1 = x_1\;(z_0z_0' + z_3z_3')\;, \\
& n_3 = x_0\;(z_0z_3' + z_3z_0')\;, \qquad n_2 = x_1\;(z_0z_3' - z_3z_0')\;; \\
(2)' & U = U_1U_3U_1' = (x_0 + ix_1\sigma_1)(z_0 + iz_3\sigma_3)(x_0' + ix_1'\sigma_1)\;, \\
& n_0 = z_0\;(x_0x_0' - x_1x_1')\;, \qquad n_3 = z_3\;(x_0x_0' + x_1x_1')\;, \\
& n_1 = z_0\;(x_0x_1' + x_1x_0')\;, \qquad -n_2 = z_3\;(x_0x_1' - x_1x_0')\;;
\end{array} \tag{108}$$

$$\begin{array}{ll}
(3) & U = U_1U_2U_1' = (x_0 + ix_1\sigma_1)(y_0 + iy_2\sigma_2)(x_0' + ix_1'\sigma_1) \\
& n_0 = y_0\;(x_0x_0' - x_1x_1')\;, \qquad n_2 = y_2\;(x_0x_0' + x_1x_1')\;, \\
& n_1 = y_0\;(x_0x_1' + x_1x_0')\;, \qquad n_3 = y_2\;(x_0x_1' - x_1x_0')\;; \\
(3)' & U = U_2U_1U_2' = (y_0 + iy_2\sigma_2)(x_0 + ix_1\sigma_1)(y_0' + iy_2'\sigma_2)\;, \\
& n_0 = x_0\;(y_0y_0' - y_2y_2')\;, \qquad n_1 = x_1\;(y_0y_0' + y_2y_2')\;, \\
& n_2 = x_0\;(y_0y_2' + y_2y_0')\;, \qquad -n_3 = x_1\;(y_0y_2' - y_2y_0')\;.
\end{array} \tag{109}$$

All results obtained can be presented in the table:

$$\begin{array}{llll}
(1) \; - & (232) & (y_0,\; y_2)\;,\; (z_0,\; z_3)\;,\; (y_0',\; y_2')\;, & (n_0,\; n_2,\; n_3,\; +n_1)\;; \\
(1)' \; - & (323) & (z_0,\; z_3)\;,\; (y_0,\; y_2)\;,\; (z_0',\; z_3')\;, & (n_0,\; n_3,\; n_2,\; -n_1)\;; \\
(2) \; - & (313) & (z_0,\; z_3)\;,\; (x_0,\; x_1)\;,\; (z_0',\; z_3')\;, & (n_0,\; n_3,\; n_1,\; +n_2)\;; \\
(2)' \; - & (131) & (x_0,\; x_1)\;,\; (z_0,\; z_3)\;,\; (x_0',\; x_1')\;, & (n_0,\; n_1,\; n_3,\; -n_2)\;; \\
(3) \; - & (121) & (x_0, x_1)\;,\; (y_0, y_2)\;,\; (x_0', x_1')\;, & (n_0,\; n_1,\; n_2,\; +n_3)\;; \\
(3)' \; - & (212) & (y_0,\; y_2)\;,\; (x_0,\; x_1)\;,\; (y_0',\; y_2')\;, & (n_0,\; n_2,\; n_1,\; -n_3)\;.
\end{array} \tag{110}$$

We see that all six factorization according to (107) – (108) – (109) have the same mathematical structure, therefore all six solutions can be produced by means of formal changes from results obtained for the case (3) - (121 – for simplicity we write down expressions for double angle variables:

$$x_0 = \cos\frac{a}{2}\ ,\ x_1 = \sin\frac{a}{2}\ ,$$

$$\cos a = \frac{(n_0 n_2 + n_1 n_3)}{\sqrt{n_0^2+n_1^2}\sqrt{n_2^2+n_3^2}}\ , \qquad \sin a = \frac{(-n_0 n_3 + n_1 n_2)}{\sqrt{n_0^2+n_1^2}\sqrt{n_2^2+n_3^2}}\ ;$$

$$y_0 = \cos\frac{b}{2}\ ,\ y_2 = \sin\frac{b}{2}\ ,$$

$$\cos b = (n_0^2+n_1^2-n_2^2-n_3^2)\ , \qquad \sin b = 2\ \sqrt{n_0^2+n_1^2}\ \sqrt{n_2^2+n_3^2}\ ;$$

$$x'_0 = \cos\frac{a'}{2}\ ,\ x'_1 = \sin\frac{a'}{2}\ ,$$

$$\cos a' = \frac{(n_0 n_2 - n_1 n_3)}{\sqrt{n_0^2+n_1^2}\sqrt{n_2^2+n_3^2}}\ , \qquad \sin a' = \frac{(n_0 n_3 + n_1 n_2)}{\sqrt{n_0^2+n_1^2}\sqrt{n_2^2+n_3^2}}\ . \tag{111}$$

3-element factorization, special case

For an arbitrary element from SU(2)

$$U = n_0 + in_1\sigma_1 + in_2\sigma_2 + in_3\sigma_3, \qquad n_0^2+n_1^2+n_2^2+n_3^2 = +1$$

let us introduce a 3-element factorization

$$U = U_1 U_2 U_3 = (x_0 + ix_1\sigma_1)(y_0 + iy_2\sigma_2)(z_0 + iz_3\sigma_3)\ ; \tag{112}$$

which gives equations

$$\begin{aligned} n_0 &= \ \ x_0y_0z_0 + x_1y_2z_3\ , \qquad & n_1 &= -x_0y_2z_3 + x_1y_0z_0\ , \\ n_2 &= \ \ x_0y_2z_0 + x_1y_0z_3\ , \qquad & n_3 &= \ \ x_0y_0z_3 - x_1y_2z_0\ . \end{aligned} \tag{113}$$

At given n_a one should find $(x_0, x_1), (y_0, y_2), (z_0, z_3)$, parameterized by angle variables as follows

$$x_0 = \cos\frac{a}{2}\ ,\ x_1 = \sin\frac{a}{2}\ , \quad y_0 = \cos\frac{b}{2}\ ,\ y_2 = \sin\frac{b}{2}\ , \quad z_0 = \cos\frac{c}{2}\ ,\ z_3 = \sin\frac{c}{2}\ .$$

Eqs. (224) can be considered as two linear systems under x_0, x_1:

$$\left\{ \begin{array}{l} \ \ x_0\ y_0z_0 + x_1\ y_2z_3 = n_0\ , \\ -x_0\ y_2z_3 + x_1\ y_0z_0 = n_1\ ; \end{array} \right. \qquad \left\{ \begin{array}{l} x_0\ y_2z_0 + x_1\ y_0z_3 = n_2\ , \\ x_0\ y_0z_3 - x_1\ y_2z_0 = n_3\ . \end{array} \right. \tag{114}$$

Their solutions are respectively:

$$\begin{aligned} x_0 &= \frac{n_0\ y_0z_0 - n_1\ y_2z_3}{y_0^2z_0^2 + y_2^2z_3^2}\ , \qquad & x_1 &= \frac{n_1\ y_0z_0 + n_0\ y_2z_3}{y_0^2z_0^2 + y_2^2z_3^2}\ ; \\ x_0 &= \frac{n_2\ y_2z_0 + n_3\ y_0z_3}{y_2^2z_0^2 + y_0^2z_3^2}\ , \qquad & x_1 &= \frac{-n_3\ y_2z_0 + n_2\ y_0z_3}{y_2^2z_0^2 + y_0^2z_3^2}\ , \end{aligned} \tag{115}$$

One can exclude variables x_0, x_1 from eqs. (115):

$$\begin{aligned} \frac{n_0y_0z_0 - n_1\ y_2z_3}{y_0^2z_0^2 + y_2^2z_3^2} &= \frac{n_2\ y_2z_0 + n_3\ y_0z_3}{y_2^2z_0^2 + y_0^2z_3^2}\ , \\ \frac{n_1\ y_0z_0 + n_0\ y_2z_3}{y_0^2z_0^2 + y_2^2z_3^2} &= \frac{-n_3\ y_2z_0 + n_2\ y_0z_3}{y_2^2z_0^2 + y_0^2z_3^2}\ . \end{aligned} \tag{116}$$

Alternatively, eqs. (224) can be considered as two linear systems under y_0, y_2:

$$\left\{ \begin{array}{l} y_0\ x_0 z_0 + y_2\ x_1 z_3 = n_0\ , \\ y_0\ x_1 z_0 - y_2\ x_0 z_3 = n_1\ ; \end{array} \right. \qquad \left\{ \begin{array}{l} y_0\ x_1 z_3 + y_2\ x_0 z_0 = n_2\ , \\ y_0\ x_0 z_3 - y_2\ x_1 z_0 = n_3\ . \end{array} \right. \tag{117}$$

Their solutions are respectively:

$$\begin{array}{ll} y_0 = \dfrac{n_0 x_0 + n_1 x_1}{z_0}\ , & y_2 = \dfrac{-n_1 x_0 + n_0 x_1}{z_3}\ ; \\ y_0 = \dfrac{n_2 x_1 + n_3 x_0}{z_3}\ , & y_2 = \dfrac{-n_3 x_1 + n_2 x_0}{z_0}\ . \end{array} \tag{118}$$

Excluding the variables y_0, y_2, we get

$$\frac{n_0 x_0 + n_1 x_1}{z_0} = \frac{n_2 x_1 + n_3 x_0}{z_3}\ , \qquad \frac{-n_1 x_0 + n_0 x_1}{z_3} = \frac{-n_3 x_1 + n_2 x_0}{z_0}\ . \tag{119}$$

Alternatively, eqs. (224) can be considered as two linear systems under z_0, z_3:

$$\left\{ \begin{array}{l} z_0\ y_0 x_0 + z_3\ y_2 x_1 = n_0\ , \\ z_0\ y_0 x_1 - z_3\ y_2 x_0 = n_1\ ; \end{array} \right. \qquad \left\{ \begin{array}{l} z_0\ y_2 x_0 + z_3\ y_0 x_1 = n_2\ , \\ -z_0\ y_2 x_1 + z_3\ y_0 x_0 = n_3\ . \end{array} \right. \tag{120}$$

Their solutions are respectively:

$$\begin{array}{ll} z_0 = \dfrac{n_0 x_0 + n_1 x_1}{y_0}\ , & z_3 = \dfrac{-n_1 x_0 + n_0 x_1}{y_2}\ ; \\ z_0 = \dfrac{n_2 x_0 - n_3 x_1}{y_2}\ , & z_3 = \dfrac{n_3 x_0 + n_2 x_1}{y_0}\ . \end{array} \tag{121}$$

Excluding the variables z_0, z_3, we get

$$\frac{n_0 x_0 + n_1 x_1}{y_0} = \frac{n_2 x_0 - n_3 x_1}{y_2}\ , \qquad \frac{-n_1 x_0 + n_0 x_1}{y_2} = \frac{n_3 x_0 + n_2 x_1}{y_0}\ . \tag{122}$$

Two last variants, (119) and (122), seem to be simpler than (116). First, let us consider the variant (119):

$$\left\{ \begin{array}{l} z_0\ (n_2 x_1 + n_3 x_0)\ -\ z_3\ (n_0 x_0 + n_1 x_1) = 0\ , \\ z_0(-n_1 x_0 + n_0 x_1) - z_3(-n_3 x_1 + n_2 x_0) = 0\ . \end{array} \right. \tag{123}$$

From vanishing the determinant

$$\left| \begin{array}{cc} (n_2 x_1 + n_3 x_0) & -(n_0 x_0 + n_1 x_1) \\ (-n_1 x_0 + n_0 x_1) & -(-n_3 x_1 + n_2 x_0) \end{array} \right| = 0$$

it follows

$$n_2 n_3\ x_1^2 - n_2^2\ x_0 x_1 + n_3^2\ x_0 x_1 - n_2 n_3\ x_0^2 - n_0 n_1\ x_0^2 + n_0^2\ x_0 x_1 - n_1^2\ x_0 x_1 + n_0 n_1\ x_1^2 = 0\ ,$$

which may be rewritten as

$$\begin{gathered} -(n_0 n_1 + n_2 n_3)(x_0^2 - x_1^2) + (n_0^2 + n_3^2 - n_1^2 - n_2^2)\ x_0 x_1 = 0\ , \\ \operatorname{tg} a = \frac{2 n_2 n_3 + 2 n_0 n_1}{n_0^2 + n_3^2 - n_1^2 - n_2^2}\ ; \end{gathered} \tag{124}$$

expressions for $\cos a$ and $\sin a$ are

$$\cos a = \frac{n_0^2 + n_3^2 - n_1^2 - n_2^2}{\sqrt{(n_0^2 + n_3^2 - n_1^2 - n_2^2)^2 + (2n_2n_3 + 2n_0n_1)^2}} \, ,$$
$$\sin a = \frac{2n_2n_3 + 2n_0n_1}{\sqrt{(n_0^2 + n_3^2 - n_1^2 - n_2^2)^2 + (2n_2n_3 + 2n_0n_1)^2}} \, . \tag{125}$$

Now, in the same manner let us consider the variant (122):

$$\left\{ \begin{array}{l} y_0 \, (n_2x_0 - n_3x_1) - y_2 \, (n_0x_0 + n_1x_1) = 0 \, , \\ y_0(-n_1x_0 + n_0x_1) - y_2(n_3x_0 + n_2x_1) = 0 \, . \end{array} \right. \tag{126}$$

From vanishing the determinant

$$\left| \begin{array}{l} (n_2x_0 - n_3x_1) - (n_0x_0 + n_1x_1) \\ (-n_1x_0 + n_0x_1) - (n_3x_0 + n_2x_1) \end{array} \right| = 0$$

it follows

$$- n_2n_3 \, x_0^2 - n_2^2 \, x_0x_1 + n_3^2 \, x_0x_1 + n_3n_2 \, x_1^2 - n_0n_1 \, x_0^2 + n_0^2 \, x_0x_1 - n_1^2 \, x_0x_1 + n_0n_1 \, x_1^2 = 0 \, ,$$

or

$$- (n_2n_3 + n_0n_1) \, (x_0^2 - x_1^2) + (n_0^2 + n_3^2 - n_1^2 - n_2^2) \, x_0x_1 = 0 \, ;$$

it may be written differently

$$- (2n_2n_3 + 2n_0n_1) \, \cos a + (n_0^2 + n_3^2 - n_1^2 - n_2^2) \, \sin a = 0 \, ,$$

which coincides with (124) as it could be expected in advance.

Turning back to (123)

$$\left\{ \begin{array}{l} z_0(n_2x_1 + n_3x_0) - z_3(n_0x_0 + n_1x_1) = 0 \, , \\ z_0(-n_1x_0 + n_0x_1) - z_3(-n_3x_1 + n_2x_0) = 0 \, . \end{array} \right.$$

we get (second equivalent variant is omitted)

$$z_0 = \cos \frac{c}{2} = \frac{n_0x_0 + n_1x_1}{\sqrt{(n_0x_0 + n_1x_1)^2 + (n_2x_1 + n_3x_0)^2}} \, ,$$
$$z_3 = \sin \frac{c}{2} = \frac{n_2x_1 + n_3x_0}{\sqrt{(n_0x_0 + n_1x_1)^2 + (n_2x_1 + n_3x_0)^2}} \, . \tag{127}$$

Let us calculate $\sin c$:

$$\sin c = 2z_0z_3 = \frac{(n_0n_2 + n_1n_3) \, \sin a + n_0n_3 \, (1 + \cos a) + n_1n_2 \, (1 - \cos a)}{(n_0x_0 + n_1x_1)^2 + (n_2x_1 + n_3x_0)^2} =$$
$$= \frac{1}{(n_0x_0 + n_1x_1)^2 + (n_2x_1 + n_3x_0)^2} \;\times$$
$$\times \; [(n_0n_3 + n_1n_2) + \frac{(n_0n_2 + n_1n_3)(2n_2n_3 + 2n_0n_1) + (n_0n_3 - n_1n_2)(n_0^2 + n_3^2 - n_1^2 - n_2^2)}{\sqrt{(n_0^2 + n_3^2 - n_1^2 - n_2^2)^2 + (2n_2n_3 + 2n_0n_1)^2}} \;] =$$
$$= \frac{(n_0n_3 + n_1n_2)}{(n_0x_0 + n_1x_1)^2 + (n_2x_1 + n_3x_0)^2} \;\; [\, 1 + \frac{1}{\sqrt{(n_0^2 + n_3^2 - n_1^2 - n_2^2)^2 + (2n_2n_3 + 2n_0n_1)^2}} \;] \, .$$

Finding expression for denominator

$$
\begin{aligned}
& (n_0x_0+n_1x_1)^2+(n_2x_1+n_3x_0)^2 = \\
&= (n_0^2+n_3^2)\frac{1+\cos a}{2}+(n_1^2+n_2^2)\frac{1-\cos a}{2}+(n_0n_1+n_2n_3)\sin a = \\
&= \frac{1}{2}\,[1+(n_0^2+n_3^2-n_1^2-n_2^2)\cos a+(2n_0n_1+2n_2n_3)\sin a] = \\
& \frac{1}{2}\,[\ 1+\sqrt{(n_0^2+n_3^2-n_1^2-n_2^2)^2+(2n_2n_3+2n_0n_1)^2}\ ] \,,
\end{aligned} \tag{128}
$$

we arrive at

$$
\sin c = \frac{2(n_0n_3+n_1n_2)}{\sqrt{(n_0^2+n_3^2-n_1^2-n_2^2)^2+(2n_2n_3+2n_0n_1)^2}} \,. \tag{129}
$$

Now let us calculate $\cos c$:

$$
\begin{aligned}
\cos c = z_0^2-z_3^2 &= \frac{(n_0^2-n_3^2)\ (1+\cos a)+(n_1^2-n_2^2)(1-\cos a)+2(n_0n_1-n_2n_3)\sin a}{2\ [\ (n_0x_0+n_1x_1)^2+(n_2x_1+n_3x_0)^2\ ]} = \\
&= [\ 1+\sqrt{(n_0^2+n_3^2-n_1^2-n_2^2)^2+(2n_2n_3+2n_0n_1)^2}\ \ ]\times \\
&\times[\ (n_0^2-n_3^2+n_1^2-n_2^2)+(n_0^2-n_3^2-n_1^2+n_2^2)\cos a+2(n_0n_1-n_2n_3)\sin a\ ] = \\
&[\ 1+\sqrt{(n_0^2+n_3^2-n_1^2-n_2^2)^2+(2n_2n_3+2n_0n_1)^2}\ ]\times \\
&\times\ (n_0^2-n_3^2+n_1^2-n_2^2)\ [\ 1+\frac{1}{\sqrt{(n_0^2+n_3^2-n_1^2-n_2^2)^2+(2n_2n_3+2n_0n_1)^2}}\ ]\,,
\end{aligned} \tag{130}
$$

that is

$$
\cos c = \frac{(n_0^2-n_3^2+n_1^2-n_2^2)}{\sqrt{(n_0^2+n_3^2-n_1^2-n_2^2)^2+(2n_2n_3+2n_0n_1)^2}} \,. \tag{131}
$$

It is a matter of simple calculation to verify the identity

$$
(n_0^2+n_3^2-n_1^2-n_2^2)^2+(2n_2n_3+2n_0n_1)^2 = (n_0^2-n_3^2+n_1^2-n_2^2)^2+(2n_0n_3+2n_1n_2)^2 \,. \tag{132}
$$

Thus, the angles a and c are determined by relations:

$$
\begin{aligned}
\cos a &= \frac{n_0^2+n_3^2-n_1^2-n_2^2}{\sqrt{(n_0^2+n_3^2-n_1^2-n_2^2)^2+(2n_2n_3+2n_0n_1)^2}}\,, \\
\sin a &= \frac{2n_2n_3+2n_0n_1}{\sqrt{(n_0^2+n_3^2-n_1^2-n_2^2)^2+(2n_2n_3+2n_0n_1)^2}}\,, \\
\cos c &= \frac{(n_0^2-n_3^2+n_1^2-n_2^2)}{\sqrt{(n_0^2-n_3^2+n_1^2-n_2^2)^2+(2n_0n_3+2n_1n_2)^2}}\,, \\
\sin c &= \frac{2(n_0n_3+n_1n_2)}{\sqrt{(n_0^2-n_3^2+n_1^2-n_2^2)^2+(2n_0n_3+2n_1n_2)^2}}\,.
\end{aligned} \tag{133}
$$

Now, turning to (126)

$$
\left\{ \begin{array}{l} y_0(n_2x_0-n_3x_1)-y_2(n_0x_0+n_1x_1)=0\,, \\ y_0(-n_1x_0+n_0x_1)-y_2(n_3x_0+n_2x_1)=0\,, \end{array} \right. \tag{134}
$$

we get expressions for y_0, y_2:

$$y_0 = \frac{(n_0 x_0 + n_1 x_1)}{\sqrt{(n_2 x_0 - n_3 x_1)^2 + (n_0 x_0 + n_1 x_1)^2}} \; ,$$
$$y_2 = \frac{(n_2 x_0 - n_3 x_1)}{\sqrt{(n_2 x_0 - n_3 x_1)^2 + (n_0 x_0 + n_1 x_1)^2}} \; . \qquad (135)$$

Let us calculate $\sin b$:

$$\sin b = 2 y_0 y_2 = \frac{n_0 n_2 (1 + \cos a) - n_1 n_3 (1 - \cos a) + (n_1 n_2 - n_0 n_3) \sin a}{(n_2 x_0 - n_3 x_1)^2 + (n_0 x_0 + n_1 x_1)^2} \; .$$

Using expression for numerator

$$n_0 n_2 (1 + \cos a) - n_1 n_3 (1 - \cos a) + (n_1 n_2 - n_0 n_3) \sin a =$$
$$= (n_0 n_2 - n_1 n_3) + (n_0 n_2 + n_1 n_3) \cos a + (n_1 n_2 - n_0 n_3) \sin a =$$
$$= (n_0 n_2 - n_1 n_3) \; [\; 1 + \frac{n_0^2 - n_3^2 + n_1^2 - n_2^2}{\sqrt{(n_0^2 + n_3^2 - n_1^2 - n_2^2)^2 + (2 n_2 n_3 + 2 n_0 n_1)^2}} \;]$$

we get

$$\sin b = \frac{(n_0 n_2 - n_1 n_3)}{(n_2 x_0 - n_3 x_1)^2 + (n_0 x_0 + n_1 x_1)^2} \; [\; 1 + \frac{n_0^2 + n_1^2 - n_2^2 - n_3^2}{\sqrt{(n_0^2 + n_3^2 - n_1^2 - n_2^2)^2 + (2 n_2 n_3 + 2 n_0 n_1)^2}} \;] \; .$$

Further, allowing for the expression for denominator

$$(n_2 x_0 - n_3 x_1)^2 + (n_0 x_0 + n_1 x_1)^2 =$$
$$= (n_2^2 + n_0^2) \frac{1 + \cos a}{2} + (n_3^2 + n_1^2) \frac{1 - \cos a}{2} + (n_0 n_1 - n_2 n_3) \sin a =$$
$$= \frac{1}{2} \; [1 + (n_2^2 + n_0^2 - n_3^2 - n_1^2) \cos a + 2 (n_0 n_1 - n_2 n_3) \sin a \;] =$$
$$= \frac{1}{2} \; [1 + \frac{n_0^2 + n_1^2 - n_2^2 - n_3^2}{\sqrt{(n_0^2 + n_3^2 - n_1^2 - n_2^2)^2 + (2 n_2 n_3 + 2 n_0 n_1)^2}} \;] \; ,$$

for $\sin b$ we obtain

$$\sin b = 2 (n_0 n_2 - n_1 n_3) \; . \qquad (136)$$

Now, let us calculate $\cos b$:

$$\cos b = y_0^2 - y_2^2 = \frac{(n_0 x_0 + n_1 x_1)^2 - (n_2 x_0 - n_3 x_1)^2}{(n_2 x_0 - n_3 x_1)^2 + (n_0 x_0 + n_1 x_1)^2} \; ,$$

Allowing for expressions for numerator

$$(n_0 x_0 + n_1 x_1)^2 - (n_2 x_0 - n_3 x_1)^2 =$$
$$= (n_0^2 - n_2^2) \frac{1 + \cos a}{2} + (n_1^2 - n_3^2) \frac{1 - \cos a}{2} + (n_0 n_1 + n_2 n_3) \sin a$$
$$= \frac{1}{2} \; [(n_0^2 - n_2^2 + n_1^2 - n_3^2) + (n_0^2 - n_2^2 - n_1^2 + n_3^2) \cos a + 2 (n_0 n_1 + n_2 n_3) \sin a \;] =$$
$$= \frac{1}{2} \; [(n_0^2 + n_1^2 - n_2^2 - n_3^2) + \sqrt{(n_0^2 + n_3^2 - n_1^2 - n_2^2)^2 + (2 n_2 n_3 + 2 n_0 n_1)^2} \;] \qquad (137)$$

and for denominator

$$
(n_2x_0 - n_3x_1)^2 + (n_0x_0 + n_1x_1)^2 =
$$

$$
= \frac{(n_0^2 + n_1^2 - n_2^2 - n_3^2) + \sqrt{(n_0^2 + n_3^2 - n_1^2 - n_2^2)^2 + (2n_2n_3 + 2n_0n_1)^2}}{2\sqrt{(n_0^2 + n_3^2 - n_1^2 - n_2^2)^2 + (2n_2n_3 + 2n_0n_1)^2}} \; ,
$$

thus $\cos b$ equals

$$
\cos b = \sqrt{(n_0^2 + n_3^2 - n_1^2 - n_2^2)^2 + (2n_2n_3 + 2n_0n_1)^2} \; . \tag{138}
$$

It is easily verified identity

$$
\begin{aligned}
\sin^2 b + \cos^2 b = 4(n_0n_2 - n_1n_3)^2 + 4(n_2n_3 + n_0n_1)^2 + (n_0^2 + n_3^2 - n_1^2 - n_2^2)^2 = \\
= 4(n_0^2 + n_3^2)(n_1^2 + n_2^2) + (n_0^2 + n_3^2 - n_1^2 - n_2^2)^2 = (n_0^2 + n_3^2 + n_1^2 + n_2^2)^2 = 1 \; .
\end{aligned}
$$

In the end of the Section let us collect obtained results: the angles a, b, c are determined by relations

$$
\begin{aligned}
\cos a &= \frac{n_0^2 + n_3^2 - n_1^2 - n_2^2}{\sqrt{(n_0^2 + n_3^2 - n_1^2 - n_2^2)^2 + (2n_2n_3 + 2n_0n_1)^2}} \; , \\
\sin a &= \frac{2n_2n_3 + 2n_0n_1}{\sqrt{(n_0^2 + n_3^2 - n_1^2 - n_2^2)^2 + (2n_2n_3 + 2n_0n_1)^2}} \; ,
\end{aligned}
$$

$$
\begin{aligned}
\cos b &= \sqrt{(n_0^2 + n_3^2 - n_1^2 - n_2^2)^2 + 4(n_2n_3 + n_0n_1)^2} = \\
&= \sqrt{1 - 4(n_0n_2 - n_1n_3)^2} \; , \qquad \sin b = 2(n_0n_2 - n_1n_3) \; ,
\end{aligned}
$$

$$
\begin{aligned}
\cos c &= \frac{(n_0^2 - n_3^2 + n_1^2 - n_2^2)}{\sqrt{(n_0^2 - n_3^2 + n_1^2 - n_2^2)^2 + (2n_0n_3 + 2n_1n_2)^2}} \; , \\
\sin c &= \frac{2(n_0n_3 + n_1n_2)}{\sqrt{(n_0^2 - n_3^2 + n_1^2 - n_2^2)^2 + (2n_0n_3 + 2n_1n_2)^2}} \; .
\end{aligned} \tag{139}
$$

They provide us with solution of the following factorization problem:

$$
U = U_1U_2U_3 = (x_0 + ix_1\sigma_1)(y_0 + iy_2\sigma_2)(z_0 + iz_3\sigma_3) \; ;
$$

$$
\begin{aligned}
x_0 = \cos(a/2) \; , \; x_1 = \sin(a/2) \; , \\
y_0 = \cos(b/2) \; , \; y_2 = \sin(b/2) \; , \\
z_0 = \cos(c/2) \; , \; z_3 = \sin(c/2) \; . \\
U = n_0 + in_1\sigma_1 + in_2\sigma_2 + in_3\sigma_3 \; ,
\end{aligned}
$$

$$
\begin{aligned}
n_0 &= \; x_0y_0z_0 + x_1y_2z_3 \; , \qquad n_1 = -x_0y_2z_3 + x_1y_0z_0 \; , \\
n_2 &= \; x_0y_2z_0 + x_1y_0z_3 \; , \qquad n_3 = \; x_0y_0z_3 - x_1y_2z_0 \; .
\end{aligned} \tag{140}
$$

All six types of 3-element factorization

There exist six different 3-element factorizations:

$$
\begin{array}{llll}
(1) & U_1U_2U_3\ , & (1)' & U_1U_3U_2\ , \\
(2) & U_2U_3U_1\ , & (2)' & U_2U_1U_3\ , \\
(3) & U_3U_1U_2\ , & (3)' & U_3U_2U_1\ .
\end{array}
\qquad (141)
$$

Let us compare six correspondent sets of equations analogues to (140):

$$
(1)-123\ , \quad n_0 = \ x_0y_0z_0 + x_1y_2z_3\ , \quad n_1 = -x_0y_2z_3 + x_1y_0z_0\ ,
$$
$$
n_2 = \ x_0y_2z_0 + x_1y_0z_3\ , \quad n_3 = \ x_0y_0z_3 - x_1y_2z_0\ ;
$$

$$
(1)'-132\ , \quad n_0 = x_0z_0y_0 + (-x_1)z_3y_2\ , \quad -n_1 = -x_0z_3y_2 + (-x_1)z_0y_0\ ,
$$
$$
n_3 = x_0z_3y_0 + (-x_1)z_0y_2\ , \quad n_2 = x_0z_0y_2 - (-x_1)z_3y_0\ ;
$$

$$
(2)-231\ , \quad n_0 = y_0z_0x_0 + y_2z_3x_1\ , \quad n_2 = -y_0z_3x_1 + y_2z_0x_0\ ,
$$
$$
n_3 = y_0z_3x_0 + y_2z_0x_1\ , \quad n_1 = y_0z_0x_1 - y_2z_3x_0\ ;
$$

$$
(2)'-213\ , \quad n_0 = y_0x_0z_0 + (-y_2)x_1z_3\ , \quad -n_2 = -y_0x_1z_3 + (-y_2)x_0z_0\ ,
$$
$$
n_1 = y_0x_1z_0 + (-y_2)x_0z_3\ , \quad n_3 = y_0x_0z_3 - (-y_2)x_1z_0\ ;
$$

$$
(3)-312\ , \quad n_0 = z_0x_0y_0 + z_3x_1y_2\ , \quad n_3 = -z_0x_1y_2 + z_3x_0y_0\ ,
$$
$$
n_1 = z_0x_1y_0 + z_3x_0y_2\ , \quad n_2 = z_0x_0y_2 - z_3x_1y_0\ ;
$$

$$
(3)'-321\ , \quad n_0 = z_0y_0x_0 + (-z_3)y_2x_1\ , \quad -n_3 = -z_0y_2x_1 + (-z_3)y_0x_0\ ,
$$
$$
n_2 = z_0y_2x_0 + (-z_3)y_0x_1\ , \quad n_1 = z_0y_0x_1 - (-z_3)y_2x_0\ ; \qquad (142)
$$

Solutions for all six problems of (142) can be obtained from the result (139) – (140) for (123)-variant with the help of formal changes described in the following table:

$$
\begin{array}{lllll}
 & a & b & c & \\
123 & (x_0,x_1) & (y_0,y_2) & (z_0,z_3) & (n_0,\ \ n_1,n_2,n_3) \\
132 & (x_0,x_1) & (z_0,z_3) & (y_0,y_2) & (n_0,-n_1,n_3,n_2) \\
231 & (y_0,y_2) & (z_0,z_3) & (x_0,x_1) & (n_0,\ \ n_2,n_3,n_1) \\
213 & (y_0,y_2) & (x_0,x_1) & (z_0,z_3) & (n_0,-n_2,n_2,n_3) \\
312 & (z_0,z_3) & (x_0,x_1) & (y_0,y_2) & (n_0,\ \ n_3,n_1,n_2) \\
321 & (z_0,z_3) & (y_0,y_2) & (x_0,x_1) & (n_0,-n_3,n_2,n_2)
\end{array}
\qquad (143)
$$

Produced formulas describing all 2-element and 3-element factorizations of the transformations of the groups SU(2) and SO(3,R) may be used in the context of the light polarization optics as a basic classification for resolving arbitrary polarization rotator into different sets of elementary ones of two or tree types:

$$\left| \begin{array}{cc} 1 & 0 \\ 0 & R \end{array} \right| \left| \begin{array}{c} I \\ I\mathbf{n} \end{array} \right| = \left| \begin{array}{c} I \\ I\mathbf{n}' \end{array} \right| .$$

6 Lorentz boosts applied to partly polarized light, Mueller formalism

Let us detail applying Lorentz transformations to describe Mueller matrices acting on a partly polarized light. Ar this we can use analogy between a Stokes 4-vector of the partly polarized light and 4-velocity vector of a massive particle in relativistic kinematics

$$S^a = (I, I\mathbf{p}) , \qquad U^a = (U^0, U^0\mathbf{V}) ; \tag{144}$$

which behave in similar manner with respect to "Lorentz" transformations:

$$A_0 = \text{ch}\frac{\beta}{2} , \; A_j = i \; \text{sh}\frac{\beta}{2} \; e_j \; , \; \mathbf{e}^2 = 1 \;\; \Longrightarrow$$
$$L = \left| \begin{array}{cc} \text{ch}\; \beta & -\mathbf{e} \; \text{sh}\; \beta \\ -\mathbf{e} \; \text{sh}\; \beta & [\delta_{ij} + (\text{ch}\; \beta - 1) e_i e_j] \end{array} \right| ;$$

this boost transformation acts on Stokes 4-vector as follows

$$I' = I \; (\text{ch}\; \beta \; - \text{sh}\; \beta \; \mathbf{ep}) \; ,$$
$$I'\mathbf{p}' = I \; [\; -\text{sh}\; \beta \; \mathbf{e} + \mathbf{p} + (\text{ch}\; \beta - 1) \; \mathbf{e}(\mathbf{ep}) \;] \; .$$

or

$$I' = I \; (\text{ch}\; \beta \; - \text{sh}\; \beta \; \mathbf{ep}) \; ,$$
$$\mathbf{p}' = \frac{-\text{sh}\; \beta \; \mathbf{e} + \mathbf{p} + (\text{ch}\; \beta - 1) \; \mathbf{e}(\mathbf{ep})}{\text{ch}\; \beta \; - \text{sh}\; \beta \; \mathbf{ep}} \; . \tag{145}$$

Several special cases should be separated.

The firs one

$$\mathbf{e} \; \mathbf{p} = 0 \; , \qquad I' = I \; \text{ch}\; \beta \; , \qquad \mathbf{p}' = \frac{-\text{sh}\; \beta \; \mathbf{e} + \mathbf{p}}{\text{ch}\; \beta} \; . \tag{146}$$

The second one

$$\underline{\mathbf{p} = +p \; \mathbf{e}} \; , \qquad I' = I \; (\text{ch}\; \beta \; - \text{sh}\; \beta \; p) \; ,$$
$$\mathbf{p}' = p' \; \mathbf{e} = \frac{-\text{sh}\; \beta + \text{ch}\; \beta \; p}{\text{ch}\; \beta \; - \text{sh}\; \beta \; p} \; \mathbf{e}. \tag{147}$$

There exists "the rest reference frame" – where the partly polarized light becomes a natural light:

$$L_0 = L_0(\beta_0, \mathbf{n}) \; , \qquad \text{th}\; \beta_0 = p \; ,$$
$$p' = 0 \; , \qquad I' = I \; (\text{ch}\; \beta_0 \; - \text{sh}_0\; \beta \; \text{th}\; \beta_0) = \frac{I}{\text{ch}\; \beta_0} \; , \tag{148}$$

The third one

$$\underline{\mathbf{p} = -p\,\mathbf{e}}\,, \qquad I' = I\;(\text{ch}\;\beta\;+\text{sh}\;\beta\;p)\;,$$
$$\mathbf{p}' = p'\;\mathbf{e} = -\frac{\text{sh}\;\beta + \text{ch}\;\beta\;p}{\text{ch}\;\beta\;+\text{sh}\;\beta\;p}\;\mathbf{e}; \tag{149}$$

here again there exists "the rest reference frame" L_0 – where the partly polarized light becomes a natural light.

Now, let us discuss the role of **relativistic ellipsoid in polarization optics**. To this end, first, let us consider a simple particular case $\mathbf{e} = (0, 0, 1)$ for eqs. (170):

$$I' = I\;(\text{ch}\;\beta\;-p_3\;\text{sh}\;\beta)\;, \qquad p'_3 = \frac{\text{ch}\;\beta p_3 - \text{sh}\;\beta}{\text{ch}\;\beta - p_3\;\text{sh}\;\beta}\;,$$
$$p'_1 = \frac{p_1}{(\text{ch}\;\beta\;-p_3\;\text{sh}\;\beta)}\;, \qquad p'_2 = \frac{p_2}{(\text{ch}\;\beta\;-p_3\;\text{sh}\;\beta)}\;. \tag{150}$$

In virtue of the main property of Lorentz transformations, we have identity

$$I'^2(1-p'^2) = I^2(1-p^2) \qquad \text{that is} \qquad 1-p'^2 = \frac{1-p^2}{(\text{ch}\;\beta\;-p_3\;\text{sh}\;\beta)^2}\;;$$

therefore the degree of polarization transforms according to the law

$$p'^2 = 1 - \frac{1-p^2}{(\text{ch}\;\beta\;-p_3\;\text{sh}\;\beta)^2}\;. \tag{151}$$

Let us express the variable p_3 through p'_3:

$$p_3 = \frac{\text{ch}\;\beta\;p'_3 + \text{sh}\;\beta}{\text{ch}\;\beta + \text{sh}\;\beta\;p'_3} \qquad \Longrightarrow \qquad \text{ch}\;\beta\;-p_3\;\text{sh}\;\beta = \frac{1}{\text{ch}\;\beta + \text{sh}\;\beta\;p'_3}\;;$$

therefore eq. (151) takes the form

$$p'^2 = 1-(1-p^2)(\text{ch}\;\beta + \text{sh}\;\beta\;p'_3)^2\;. \tag{152}$$

It remains to shows that eq. (152) represents an ellipsoid. Indeed, it reads

$$p'^2_1 + p'^2_2 + p'^2_3 + (1-p^2) 2\;\text{ch}\;\beta\;\text{sh}\beta\;p'_3 + (1-p^2)\text{sh}^2\beta\;p'^2_3 = 1-(1-p^2)\text{ch}^2\beta\;,$$

or

$$p'^2_1 + p'^2_2 + (\text{ch}^2\beta - p^2\;\text{sh}^2\beta)\;\left[p'_3 + \frac{(1-p^2)\text{sh}\;\beta\;\text{ch}\;\beta}{\text{ch}^2\beta - p^2\;\text{sh}^2\beta}\right]^2 =$$
$$= p^2\text{ch}^2\beta - \text{sh}^2\beta\;+\frac{(1-p^2)^2\text{sh}^2\beta\;\text{ch}^2\beta}{\text{ch}^2\beta - p^2\;\text{sh}^2\beta}\;,$$

and finally we arrive at the equation

$$p'^2_1 + p'^2_2 + (\text{ch}^2\beta - p^2\;\text{sh}^2\beta)\;(p'_3 + \gamma)^2 = \frac{p^2}{\text{ch}^2\beta - p^2\text{sh}^2\beta}\;, \tag{153}$$

where

$$\gamma = \frac{(1-p^2)\,\text{sh}\,\beta\;\text{ch}\,\beta}{\text{ch}^2\beta - p^2\,\text{sh}^2\beta}\;, \qquad \text{ch}^2\beta - p^2\,\text{sh}^2\beta = \text{ch}^2\beta(1-p^2) + p^2 > 0\;.$$

Equation (153) describes an ellipsoid.

These results can easily be extended to a general case of arbitrary transformation L:

$$I' = I\;(\text{ch}\,\beta - \text{sh}\,\beta\;(\mathbf{ep}))\;, \qquad \mathbf{p}' = \frac{\mathbf{p} - \mathbf{e}\,\text{sh}\,\beta + (\text{ch}\,\beta - 1)\;\mathbf{e}\;(\mathbf{ep})}{\text{ch}\,\beta - \text{sh}\,\beta\;\mathbf{e}\;\mathbf{p}}\;.$$

Again, one can apply the identity

$$I'^2(1-p'^2) = I^2(1-p^2) \qquad \Longrightarrow \qquad 1-p'^2 = \frac{1-p^2}{[\text{ch}\,\beta - \text{sh}\,\beta\;(\mathbf{ep})]^2}\;, \tag{154}$$

and exclude the variable $\mathbf{p}$:

$$\mathbf{p} = \frac{\mathbf{p}' + \mathbf{e}\,\text{sh}\,\beta + (\text{ch}\,\beta - 1)\;\mathbf{e}(\mathbf{e}\;\mathbf{p}')}{\text{ch}\,\beta + \text{sh}\,\beta\;\mathbf{e}\;\mathbf{p}'}\;,$$

and

$$\text{ch}\,\beta - \text{sh}\,\beta(\mathbf{ep}) = \frac{1}{\text{ch}\,\beta + \text{sh}\,\beta\;\mathbf{ep}'} \tag{155}$$

Thus, eq. (154) will takes the form

$$1-p'^2 = (1-p^2)\;(\text{ch}\,\beta + \text{sh}\,\beta\;\mathbf{ep}')^2\;; \tag{156}$$

this is an equation of an ellipsoid oriented along the vector $\mathbf{e}$.

7 On small Lorentz group for time-like vectors of a partly polarized light

Now let us specify the known in relativistic kinematics problem of stationary Lorentz subgroup in the above parametrization and in the context of polarization optics:

$$L_b{}^a(k,\bar{k}^*)\;S_a = +S_b\;, \qquad S^a S_a = \text{inv} > 0\;; \tag{157}$$

or taking into account relations (223) $L = \hat{A}\hat{A}^* = \hat{A}^*\hat{A}$, eq. (157) reads

$$\hat{A}\;S = (\hat{A}^*)^{-1}\;S \qquad \Longrightarrow \qquad [\hat{A} - (\hat{A}^*)^{-1}\;]\;S = 0\;, \tag{158}$$

where

$$\hat{A} = \left|\begin{array}{rrrr} k_0 & -k_1 & -k_2 & -k_3 \\ -k_1 & k_0 & -ik_3 & ik_2 \\ -k_2 & ik_3 & k_0 & -ik_1 \\ -k_3 & -ik_2 & ik_1 & k_0 \end{array}\right|\;,\;(\hat{A}^*)^{-1} = \left|\begin{array}{rrrr} k_0^* & k_1^* & k_2^* & k_3^* \\ k_1^* & k_0^* & -ik_3^* & ik_2^* \\ k_2^* & ik_3^* & k_0^* & -ik_1^* \\ k_3^* & -ik_2^* & ik_1^* & k_0^* \end{array}\right|\;.$$

Therefore, eq. (158) takes the form

$$\left| \begin{array}{cccc} (k_0 - k_0^*) & -(k_1 + k_1^*) & -(k_2 + k_2^*) & -(k_3 + k_3^*) \\ -(k_1 + k_1^*) & (k_0 - k_0^*) & -i(k_3 - k_3^*) & i(k_2 - k_2^*) \\ -(k_2 + k_2^*) & i(k_3 - k_3^*) & (k_0 - k_0^*) & -i(k_1 - k_1^*) \\ -(k_3 + k_3^*) & -i(k_2 - k_2^*) & i(k_1 - k_1^*) & (k_0 - k_0^*) \end{array} \right| \left| \begin{array}{c} S_0 \\ S_1 \\ S_2 \\ S_3 \end{array} \right| = 0 \qquad (159)$$

which with the use of notation

$$k_0 = n_0 + im_0 \, , \qquad k_j = -in_j + m_j$$

reads

$$\left| \begin{array}{cccc} im_0 & -m_1 & -m_2 & -m_3 \\ -m_1 & im_0 & -n_3 & n_2 \\ -m_2 & n_3 & im_0 & -n_1 \\ -m_3 & -n_2 & n_1 & im_0 \end{array} \right| \left| \begin{array}{c} S_0 \\ S_1 \\ S_2 \\ S_3 \end{array} \right| = 0 \qquad (160)$$

It may be noted that when imposing restriction $m_0 = 0, m_j = 0$ we arrive at

$$\left| \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & -n_3 & n_2 \\ 0 & n_3 & 0 & -n_1 \\ 0 & -n_2 & n_1 & 0 \end{array} \right| \left| \begin{array}{c} S_0 \\ S_1 \\ S_2 \\ S_3 \end{array} \right| = 0 \qquad \Longrightarrow \qquad \mathbf{n} = \frac{\mathbf{S}}{S} \qquad (161)$$

which corresponds to rotational 1-parametric group $O(\phi, \mathbf{n})$ leaving invariant vector $\mathbf{S} = S\mathbf{n}$.

In general case eq. (160) can be rewritten as follows

$$\begin{aligned} im_0 S_0 - m_1 S_1 - m_2 S_2 - m_3 S_3 = 0 \\ -m_1 S_0 + im_0 S_1 - n_3 S_2 + n_2 S_3 = 0 \\ -m_2 S_0 + n_3 S_1 + im_0 S_2 - n_1 S_3 = 0 \\ -m_3 S_0 - n_2 S_1 + n_1 S_2 + im_0 S_3 = 0 \end{aligned}$$

To satisfy these 4 equations one must require identity $m_0 \equiv 0$, then

$$\begin{aligned} m_1 S_1 + m_2 S_2 + m_3 S_3 = 0 \\ -m_1 S_0 - n_3 S_2 + n_2 S_3 = 0 \\ -m_2 S_0 + n_3 S_1 - n_1 S_3 = 0 \\ -m_3 S_0 - n_2 S_1 + n_1 S_2 = 0 \, . \end{aligned} \qquad (162)$$

It is easily seen that the first equation can be derived from 3 remaining ones, so we have only three independent equations making linear system under n_1, n_2, n_3:

$$\begin{aligned} 0 \; + S_3 \, n_2 - S_2 \, n_3 = m_1 \, S_0 \\ -S_3 \, n_1 + 0 \; + S_1 \, n_3 = m_2 \, S_0 \\ S_2 \, n_1 - S_1 \, n_2 + 0 \; = m_3 \, S_0 \end{aligned} \qquad (163)$$

This system can be resolved under m_j (remembering on $p_j = S_j / S_0$):

$$m_1 = p_3 \, n_2 - p_2 \, n_3 \; , \qquad m_2 = -p_3 \, n_1 + p_1 \, n_3 \; ,$$
$$m_3 = p_2 \, n_1 - p_1 \, n_2 \; , \qquad \text{that is} \qquad \mathbf{m} = \mathbf{n} \times \mathbf{p} \; . \tag{164}$$

Thus, the problem of stationary subgroup is solved (first it was done by Winer [...])

$$L_b{}^a(k, \bar{k}^*) \; S_a = +S_b \; , \qquad S^a S_a = \text{inv} > 0 \; ;$$
$$k_0 = n_0 + i0 \; , \qquad \mathbf{k} = -i \; \mathbf{n} + \mathbf{n} \times \mathbf{p} \; . \tag{165}$$

There exist special and simplest case when $\mathbf{n} \uparrow\uparrow \mathbf{p}$: then $\mathbf{m} = 0$ and we arrive at a 1-parametric subgroup in $SO(3.R)$.

There exist simple considerations that help to have understood more on the structure of the above small (or stationary) subgroup. They are based on the use of a Lorentz transformation to a rest reference frame. Indeed,

$$S = (S_0, \mathbf{S}) = (S_0, S_0 p \; \mathbf{n}) \; , \qquad S^{rest} = (S_0^{rest}, 0, 0, 0) \; ,$$
$$L_{rest} = O \; L_0(\beta_0, \mathbf{n}) \; , \; , \qquad O \; L_0 \; S = S_{rest} \; , \tag{166}$$

In the rest reference frame the problem of stationary subgroup can be solved straightforwardly:

$$L_{stat} = O' \; , \qquad O' S_{rest} = S_{rest} \qquad \Longrightarrow$$
$$O'O \; L_0 S = O \; L_0 \; S \qquad \Longrightarrow \qquad L_0^{-1} \; [O^{-1} O' O] \; L_0 S = \; S \tag{167}$$

therefore in any reference frame the stationary Lorentz subgroup is isomorphic to a rotation group (realized by matrices O''):

$$L_{stat} = L_0^{-1} \; [\; O^{-1} O' O \;] \; L_0 = L_0^{-1} \; O'' \; L_0 \tag{168}$$

8 Lorentz boosts applied to a completely polarized light

Let us detail applying Lorentz transformations to describe Mueller matrices acting on a completely polarized light. Ar this a Stokes 4-vector is

$$S^a = (I, I\mathbf{n}) \; , \qquad \mathbf{n}^2 = 1 \; ; \tag{169}$$

which under "Lorentz boost"

$$L = \left| \begin{array}{cc} \text{ch} \; \beta & -\mathbf{e} \; \text{sh} \; \beta \\ -\mathbf{e} \; \text{sh} \; \beta & [\delta_{ij} + (\text{ch} \; \beta - 1) e_i e_j] \end{array} \right| ;$$

this boost transformation acts on Stokes 4-vector as follows

$$I' = I \; (\text{ch} \; \beta \; - \text{sh} \; \beta \; \mathbf{en}) \; , \qquad \mathbf{n}' = \frac{-\text{sh} \; \beta \; \mathbf{e} + \mathbf{n} + (\text{ch} \; \beta - 1) \; \mathbf{e}(\mathbf{en})}{\text{ch} \; \beta \; - \text{sh} \; \beta \; \mathbf{en}} \; . \tag{170}$$

Several special cases immediately is seen.

The first one:

$$\underline{\mathbf{e}\,\mathbf{n} = 0}\,, \qquad I' = I \;\; \mathrm{ch}\,\beta\,, \qquad \mathbf{n}' = \frac{\mathbf{n} - \mathrm{sh}\,\beta\,\mathbf{e}}{\mathrm{ch}\,\beta} \tag{171}$$

The second one:

$$\begin{gathered} \underline{\mathbf{e} = +\,\mathbf{n}}\,, \qquad I' = I\,(\mathrm{ch}\,\beta\; - \mathrm{sh}\,\beta\,) = I\,e^{-\beta}\,, \\ \mathbf{n}' = \frac{-\mathrm{sh}\,\beta\,\mathbf{n} + \mathbf{n} + (\mathrm{ch}\,\beta - 1)\,\mathbf{n}}{\mathrm{ch}\,\beta\; - \mathrm{sh}\,\beta} = +\mathbf{n} \end{gathered} \tag{172}$$

and similar the third one:

$$\begin{gathered} \underline{\mathbf{e} = -\,\mathbf{n}}\,, \qquad I' = I\,(\mathrm{ch}\,\beta\; + \mathrm{sh}\,\beta\,) = I\,e^{+\beta}\,, \\ \mathbf{n}' = \frac{\mathrm{sh}\,\beta\,\mathbf{n} + \mathbf{n} + (\mathrm{ch}\,\beta - 1)\,\mathbf{n}}{\mathrm{ch}\,\beta\; + \mathrm{sh}\,\beta} = +\mathbf{n} \end{gathered} \tag{173}$$

9 On small Lorentz group for isotropic vectors of a completely polarized light

Now let us specify the known in relativistic kinematics problem of stationary Lorentz subgroup in the above parametrization and in the context of polarization optics:

$$L_b{}^a(k, \bar{k}^*)\,S_a = +S_b\,, \qquad S^a S_a = 0\,; \tag{174}$$

with the help of the factorization $L = \hat{A}\hat{A}^* = \hat{A}^*\hat{A}$,

$$\hat{A}\,S = (\hat{A}^*)^{-1}\,S \qquad \Longrightarrow \qquad [\hat{A} - (\hat{A}^*)^{-1}\,]\,S = 0\,, \tag{175}$$

and further we get

$$\left|\begin{array}{rrrr} (k_0 - k_0^*) & -(k_1 + k_1^*) & -(k_2 + k_2^*) & -(k_3 + k_3^*) \\ -(k_1 + k_1^*) & (k_0 - k_0^*) & -i(k_3 - k_3^*) & i(k_2 - k_2^*) \\ -(k_2 + k_2^*) & i(k_3 - k_3^*) & (k_0 - k_0^*) & -i(k_1 - k_1^*) \\ -(k_3 + k_3^*) & -i(k_2 - k_2^*) & i(k_1 - k_1^*) & (k_0 - k_0^*) \end{array}\right| \left|\begin{array}{c} I \\ Ip_1 \\ Ip_2 \\ Ip_3 \end{array}\right| = 0\,, \tag{176}$$

or

$$\begin{gathered} k_0 = n_0 + im_0\,, \qquad\qquad k_j = -in_j + m_j \\ \left|\begin{array}{rrrr} im_0 & -m_1 & -m_2 & -m_3 \\ -m_1 & im_0 & -n_3 & n_2 \\ -m_2 & n_3 & im_0 & -n_1 \\ -m_3 & -n_2 & n_1 & im_0 \end{array}\right| \left|\begin{array}{c} I \\ Ip_1 \\ Ip_2 \\ Ip_3 \end{array}\right| = 0\,, \qquad \mathbf{p}^2 = 1\,. \end{gathered} \tag{177}$$

When imposing restriction $m_0 = 0, m_j = 0$ we arrive at

$$\left|\begin{array}{rrrr} 0 & 0 & 0 & 0 \\ 0 & 0 & -n_3 & n_2 \\ 0 & n_3 & 0 & -n_1 \\ 0 & -n_2 & n_1 & 0 \end{array}\right| \left|\begin{array}{c} I \\ Ip_1 \\ Ip_2 \\ Ip_3 \end{array}\right| = 0 \qquad \Longrightarrow \qquad \mathbf{n} = \frac{\mathbf{S}}{S} \tag{178}$$

which corresponds to rotational 1-parametric group $O(\phi, \mathbf{p})$ leaving invariant vector $\mathbf{p}$.

In general case eq. (160) can be rewritten as follows (one must require identity $m_0 \equiv 0$)

$$
\begin{aligned}
m_1 p_1 + m_2 p_2 + m_3 p_3 = 0 \\
-m_1 - n_3 p_2 + n_2 p_3 = 0 \\
-m_2 + n_3 p_1 - n_1 p_3 = 0 \\
-m_3 - n_2 p_1 + n_1 p_2 = 0 \; .
\end{aligned} \tag{179}
$$

The first equation can be derived from 3 remaining ones, so we have only three independent equations

$$
\begin{aligned}
m_1 = p_3\, n_2 - p_2\, n_3 \; , \qquad m_2 = -p_3\, n_1 + p_1\, n_3 \; , \\
m_3 = p_2\, n_1 - p_1\, n_2 \; , \qquad \text{that is} \qquad \mathbf{m} = \mathbf{n} \times \mathbf{p} \; .
\end{aligned} \tag{180}
$$

Thus, solution of the problem of stationary subgroup is given by the same relations as in time-like case:

$$
\begin{aligned}
L_b{}^a(k, \bar{k}^*)\; S_a = +S_b \; , \qquad S^a S_a = 0 \; ; \\
k_0 = n_0 + i\, 0 \; , \qquad \mathbf{k} = -i\, \mathbf{n} + \mathbf{n} \times \mathbf{p} \; .
\end{aligned} \tag{181}
$$

There exist special and simplest case when $\mathbf{n} \uparrow\uparrow \mathbf{p}$: then $\mathbf{m} = 0$ and we arrive at 1-parametric subgroup in $SO(3.R)$.

10 On transitivity problem $LS = S'$ in polarization optics

Now let us specify the known in relativistic kinematics the transitivity problem in the above parametrization and in the context of polarization optics

$$
L_b{}^a(k, \bar{k}^*)\; S_a = +S'_b \; . \tag{182}
$$

From the very beginning, it should be mentioned on some peculiarities in the problem. Indeed, if Lorentz matrix a Lorentz transitivity matrix L obeys the equation $LS = S'$, then $L\,(L_{stat} S) = L'_{stat} S'$, and therefore

$$
\begin{aligned}
[\; (L'_{stat})^{-1} L\; L_{stat}\;]\; S = S' \qquad \Longrightarrow \\
(L'_{stat})^{-1} L\; L_{stat} \qquad \text{is a transitivity matrix as well}
\end{aligned} \tag{183}
$$

In other words, in general, a transitivity transformation $L\;S = S'$ is determined with a 6-parametric freedom. The later should be taken into account when searching for explicit representation for a transitivity matrix.

With the use of factorized representation of the Lorentz matrices, the problem reduces to

$$
\hat{A}^* S = A^{-1} S' \; , \qquad \text{and} \qquad \hat{A}\; S = (\hat{A}^*)^{-1}\; S' \; , \tag{184}
$$

so that

$$\left| \begin{array}{rrrr} k_0^* & -k_1^* & -k_2^* & -k_3^* \\ -k_1^* & k_0^* & ik_3^* & -ik_2^* \\ -k_2^* & -ik_3^* & k_0^* & ik_1^* \\ -k_3^* & ik_2^* & -ik_1^* & k_0^* \end{array} \right| \left| \begin{array}{c} S_0 \\ S_1 \\ S_2 \\ S_3 \end{array} \right| = \left| \begin{array}{rrrr} k_0 & k_1 & k_2 & k_3 \\ k_1 & k_0 & ik_3 & -ik_2 \\ k_2 & -ik_3 & k_0 & ik_1 \\ k_3 & ik_2 & -ik_1 & k_0 \end{array} \right| \left| \begin{array}{c} S_0' \\ S_1' \\ S_2' \\ S_3' \end{array} \right| ,$$

and

$$\left| \begin{array}{rrrr} k_0 & -k_1 & -k_2 & -k_3 \\ -k_1 & k_0 & -ik_3 & ik_2 \\ -k_2 & ik_3 & k_0 & -ik_1 \\ -k_3 & -ik_2 & ik_1 & k_0 \end{array} \right| \left| \begin{array}{c} S_0 \\ S_1 \\ S_2 \\ S_3 \end{array} \right| = \left| \begin{array}{rrrr} k_0^* & k_1^* & k_2^* & k_3^* \\ k_1^* & k_0^* & -ik_3^* & ik_2^* \\ k_2^* & ik_3^* & k_0^* & -ik_1^* \\ k_3^* & -ik_2^* & ik_1^* & k_0^* \end{array} \right| \left| \begin{array}{c} S_0' \\ S_1' \\ S_2' \\ S_3' \end{array} \right| . \qquad (185)$$

Summing and subtracting equations we get (remembering $k_0 = n_0 + im_0$, $k_j = -in_j + m_j$)

$$\left| \begin{array}{rrrr} n_0 & -m_1 & -m_2 & -m_3 \\ -m_1 & n_0 & -n_3 & n_2 \\ -m_2 & n_3 & n_0 & -n_1 \\ -m_3 & -n_2 & n_1 & n_0 \end{array} \right| \left| \begin{array}{c} S_0 \\ S_1 \\ S_2 \\ S_3 \end{array} \right| = \left| \begin{array}{rrrr} n_0 & m_1 & m_2 & m_3 \\ m_1 & n_0 & n_3 & -n_2 \\ m_2 & -n_3 & n_0 & n_1 \\ m_3 & n_2 & -n_1 & n_0 \end{array} \right| \left| \begin{array}{c} S_0' \\ S_1' \\ S_2' \\ S_3' \end{array} \right| ,$$

$$\left| \begin{array}{rrrr} -m_0 & -n_1 & -n_2 & -n_3 \\ -n_1 & -m_0 & m_3 & -m_2 \\ -n_2 & -m_3 & -m_0 & m_1 \\ -n_3 & m_2 & -m_1 & -m_0 \end{array} \right| \left| \begin{array}{c} S_0 \\ S_1 \\ S_2 \\ S_3 \end{array} \right| = \left| \begin{array}{rrrr} m_0 & -n_1 & -n_2 & -n_3 \\ -n_1 & m_0 & m_3 & -m_2 \\ -n_2 & -m_3 & m_0 & m_1 \\ -n_3 & m_2 & -m_1 & m_0 \end{array} \right| \left| \begin{array}{c} S_0' \\ S_1' \\ S_2' \\ S_3' \end{array} \right| .$$

These give linear equations:

$$\begin{array}{l} n_0\,(S_0 - S_0') - m_1\,(S_1 + S_1') - m_2\,(S_2 + S_2') - m_3\,(S_3 + S_3') = 0 \,, \\ -m_1\,(S_0 + S_0') + n_0\,(S_1 - S_1') + n_2\,(S_3 + S_3') - n_3\,(S_2 + S_2') = 0 \,, \\ -m_2\,(S_0 + S_0') + n_0\,(S_2 - S_2') + n_3\,(S_1 + S_1') - n_1\,(S_3 + S_3') = 0 \,, \\ -m_3\,(S_0 + S_0') + n_0\,(S_3 - S_3') + n_1\,(S_2 + S_2') - n_2\,(S_1 + S_1') = 0 \,, \end{array}$$

$$\begin{array}{l} -\,m_0\,(S_0 + S_0') - n_1\,(S_1 - S_1') - n_2\,(S_2 - S_2') - n_3\,(S_3 - S_3') = 0 \,, \\ -n_1\,(S_0 - S_0') - m_0\,(S_1 + S_1') - m_2\,(S_3 - S_3') + m_3\,(S_2 - S_2') = 0 \,, \\ -n_2\,(S_0 - S_0') - m_0\,(S_2 + S_2') - m_3\,(S_1 - S_1') + m_1\,(S_3 - S_3') = 0 \,, \\ -n_3\,(S_0 - S_0') - m_0\,(S_3 + S_3') - m_1\,(S_2 - S_2') + m_2\,(S_1 - S_1') = 0 \,. \end{array} \qquad (186)$$

For a special (non-relativistic) case $S_0' = S_0 = I$, the system (**??**) becomes simpler (also let us impose additional restrictions $m_0 = 0,\ m_j = 0$):

$$\begin{array}{l} n_0\,(S_1 - S_1') + n_2\,(S_3 + S_3') - n_3\,(S_2 + S_2') = 0 \,, \\ n_0\,(S_2 - S_2') + n_3\,(S_1 + S_1') - n_1\,(S_3 + S_3') = 0 \,, \\ n_0\,(S_3 - S_3') + n_1\,(S_2 + S_2') - n_2\,(S_1 + S_1') = 0 \,, \\ -n_1\,(S_1 - S_1') - n_2\,(S_2 - S_2') - n_3\,(S_3 - S_3') = 0 \,, \end{array} \qquad (187)$$

Here the fourth equation follows from three first:

$$
\begin{aligned}
n_2\,(S_3+S'_3)-n_3\,(S_2+S'_2) &= -n_0\,(S_1-S'_1)\,,\\
n_3\,(S_1+S'_1)-n_1\,(S_3+S'_3) &= -n_0\,(S_2-S'_2)\,,\\
n_1\,(S_2+S'_2)-n_2\,(S_1+S'_1) &= -n_0\,(S_3-S'_3)\,,
\end{aligned}
$$

or in vector form

$$
\mathbf{n}\times(\mathbf{S}+\mathbf{S}') = -n_0\,(\mathbf{S}-\mathbf{S}')\,. \tag{188}
$$

Solution can be constructed with the help of the substitution $\mathbf{n}=\alpha\,\mathbf{S}+N_-\,\mathbf{S}'+\gamma\,\mathbf{S}\times\mathbf{S}'$, then

$$
\begin{aligned}
&(\alpha-N_-)\,\mathbf{S}\times\mathbf{S}'+\gamma\,[\,\mathbf{S}'\,S^2+\mathbf{S}'\,(\mathbf{S}\mathbf{S}')-\mathbf{S}\,S^2-\mathbf{S}\,(\mathbf{S}\mathbf{S}')\,] =\\
&= -n_0\mathbf{S}+n_0\mathbf{S}' \qquad \Longrightarrow \qquad N_-=\alpha\,, \qquad n_0=\gamma\,(S^2+\mathbf{S}\,\mathbf{S}')\,;
\end{aligned}
$$

therefore

$$
\mathbf{n}=\alpha\,(\mathbf{S}+\mathbf{S}')+n_0\,\frac{\mathbf{S}\times\mathbf{S}'}{S^2+\mathbf{S}\,\mathbf{S}'}\,, \qquad \alpha \ \text{ is arbitrary}\,. \tag{189}
$$

The most simple result is when an arbitrary parameter α vanishes:

$$
\begin{aligned}
&\mathbf{n}=n_0\,\frac{\mathbf{S}\times\mathbf{S}'}{S^2+\mathbf{S}\,\mathbf{S}'}\,, \qquad \mathbf{S}\mathbf{S}'=S^2\cos\phi\,,\\
&\mathbf{S}\times\mathbf{S}'=S^2\,\sin\phi\,\mathbf{n}_0\,, \qquad n_0=\cos\frac{\phi}{2}\,, \qquad \mathbf{n}=\cos\frac{\phi}{2}\,\mathbf{n}_0\,,
\end{aligned} \tag{190}
$$

what is well-known and evident from geometry consideration.

Let us turn to the general (relativistic) case (186):

$$
\begin{aligned}
m_1\,(S_1+S'_1)+m_2\,(S_2+S'_2)+m_3\,(S_3+S'_3) &= n_0\,(S_0-S'_0)\,,\\
m_1\,(S_0+S'_0)-n_2\,(S_3+S'_3)+n_3\,(S_2+S'_2) &= n_0\,(S_1-S'_1)\,,\\
m_2\,(S_0+S'_0)-n_3\,(S_1+S'_1)+n_1\,(S_3+S'_3) &= n_0\,(S_2-S'_2)\,,\\
m_3\,(S_0+S'_0)-n_1\,(S_2+S'_2)+n_2\,(S_1+S'_1) &= n_0\,(S_3-S'_3)\,,\\
\\
-\,n_1\,(S_1-S'_1)-n_2\,(S_2-S'_2)-n_3\,(S_3-S'_3) &= m_0\,(S_0+S'_0)\,,\\
-n_1\,(S_0-S'_0)-m_2\,(S_3-S'_3)+m_3\,(S_2-S'_2) &= m_0\,(S_1+S'_1)\,,\\
-n_2\,(S_0-S'_0)-m_3\,(S_1-S'_1)+m_1\,(S_3-S'_3) &= m_0\,(S_2+S'_2)\,,\\
-n_3\,(S_0-S'_0)-m_1\,(S_2-S'_2)+m_2\,(S_1-S'_1) &= m_0\,(S_3+S'_3)\,.
\end{aligned} \tag{191}
$$

restrictions must hold

$$
n_0^2+\mathbf{n}^2-m_0^2-\mathbf{m}^2=1\,, \qquad n_0m_0+\mathbf{n}\mathbf{m}=0\,; \tag{192}
$$

therefore existing trivial solution $n_a=0, m_a=0$ of eqs. (191) has no interest. Eqs. (191) may be rewritten in vector form:

$$
\begin{aligned}
&(1) \qquad \mathbf{m}\,(\mathbf{S}+\mathbf{S}') = n_0\,(S_0-S'_0)\,,\\
&(2) \qquad \mathbf{n}\,(\mathbf{S}-\mathbf{S}') = -m_0\,(S_0+S'_0)\,,\\
&(3) \qquad \mathbf{m}\,(S_0+S'_0)+(\mathbf{S}+\mathbf{S}')\times\mathbf{n} = n_0\,(\mathbf{S}-\mathbf{S}')\,,\\
&(4) \qquad \mathbf{n}\,(S_0-S'_0)-(\mathbf{S}-\mathbf{S}')\times\mathbf{m} = -m_0\,(\mathbf{S}+\mathbf{S}')\,.
\end{aligned} \tag{193}
$$

It must be noted that non-relativity restriction $S_0 - S_0' = 0$ immediately leads to relations $\mathbf{m} = 0$ and $m_0 = 0$.

Below we will presuppose a relativistic case only, when excluding the variables m_0, n_0 in eqs. (193) is possible:

$$n_0 = \frac{\mathbf{m}\,(\mathbf{S} + \mathbf{S}')}{S_0 - S_0'}\;, \qquad m_0 = -\frac{\mathbf{n}\,(\mathbf{S} - \mathbf{S}')}{S_0 + S_0'}\;\;; \tag{194}$$

remaining equations are

$$\begin{gathered}\mathbf{m}\,(S_0 + S_0') + (\mathbf{S} + \mathbf{S}') \times \mathbf{n} = \frac{\mathbf{m}\,(\mathbf{S} + \mathbf{S}')}{S_0 - S_0'}\;\,(\mathbf{S} - \mathbf{S}')\;, \\ \mathbf{n}\,(S_0 - S_0') - (\mathbf{S} - \mathbf{S}') \times \mathbf{m} = \frac{\mathbf{n}\,(\mathbf{S} - \mathbf{S}')}{S_0 + S_0'}\;\,(\mathbf{S} + \mathbf{S}')\;. \end{gathered} \tag{195}$$

Let us introduce notation $S_0 \pm S_0' = S_0^{\pm}$, $\mathbf{S} \pm \mathbf{S}' = \mathbf{S}^{\pm}$, and substitutions:

$$\begin{gathered}\mathbf{n} = N_+\,\mathbf{S}^+ + N_-\,\mathbf{S}^- + N\;\;\mathbf{S}^- \times \mathbf{S}^+\;, \\ \mathbf{m} = M_+\,\mathbf{S}^+ + M_-\,\mathbf{S}^- + M\;\;\mathbf{S}^- \times \mathbf{S}^+\;, \end{gathered} \tag{196}$$

equations (195) take the form

$$\begin{aligned}&(M_+\,\mathbf{S}^+ + M_-\,\mathbf{S}^- + M\;\;\mathbf{S}^- \times \mathbf{S}^+)\;S_0^+ + \mathbf{S}^+ \times (N_+\,\mathbf{S}^+ + N_-\,\mathbf{S}^- + N\;\;\mathbf{S}^- \times \mathbf{S}^+) = \\ &\qquad = [(M_+\,\mathbf{S}^+ + M_-\,\mathbf{S}^- + M\;\;\mathbf{S}^- \times \mathbf{S}^+)\;\mathbf{S}^+]\;\frac{\mathbf{S}^-}{S_0^-}\;,\end{aligned}$$

$$\begin{aligned}&(N_+\,\mathbf{S}^+ + N_-\,\mathbf{S}^- + N\;\;\mathbf{S}^- \times \mathbf{S}^+)\;S_0^- - \mathbf{S}^- \times (M_+\,\mathbf{S}^+ + M_-\,\mathbf{S}^- + M\;\;\mathbf{S}^- \times \mathbf{S}^+) = \\ &\qquad = [(N_+\,\mathbf{S}^+ + N_-\,\mathbf{S}^- + N\;\;\mathbf{S}^- \times \mathbf{S}^+)\;\mathbf{S}^-]\;\frac{\mathbf{S}^+}{S_0^+}\;.\end{aligned}$$

or

$$\begin{aligned}&S_0^+\;(M_+\,\mathbf{S}^+ + M_-\,\mathbf{S}^- + M\;\;\mathbf{S}^- \times \mathbf{S}^+) - N_-\;(\mathbf{S}^- \times \mathbf{S}^+) + N\;[\;\mathbf{S}^-\;(\mathbf{S}^+\mathbf{S}^+) - \mathbf{S}^+\;(\mathbf{S}^+\mathbf{S}^-)\;] = \\ &\qquad = [M_+\;(\mathbf{S}^+\mathbf{S}^+) + M_-\;(\mathbf{S}^-\;\mathbf{S}^+)\;]\;\frac{\mathbf{S}^-}{S_0^-}\;,\end{aligned}$$

$$\begin{aligned}&S_0^-\;(N_+\,\mathbf{S}^+ + N_-\,\mathbf{S}^- + N\;\;\mathbf{S}^- \times \mathbf{S}^+) - M_+\;(\mathbf{S}^- \times \mathbf{S}^+) - M\;[\;\mathbf{S}^-\;(\mathbf{S}^-\mathbf{S}^+) - \mathbf{S}^+\;(\mathbf{S}^-\mathbf{S}^-)\;] = \\ &\qquad = [\;N_+\;(\mathbf{S}^+\mathbf{S}^-) + N_-\;(\mathbf{S}^-\mathbf{S}^-)\;]\;\frac{\mathbf{S}^+}{S_0^+}\;.\end{aligned}$$

Further, we arrive at 6 relations on parameters $N_{\pm}, N, M_{\pm}, M$:

$$\begin{aligned}M_+\;S_0^+ - N\;(\mathbf{S}^+\mathbf{S}^-) &= 0\;, \\ N_-\;S_0^- - M\;(\mathbf{S}^-\mathbf{S}^+) &= 0\;, \\ M\;S_0^+ - N_- &= 0\;, \\ N\;S_0^- - M_+ &= 0\;,\end{aligned} \tag{197}$$

$$\begin{aligned}M_-\;S_0^+ + N\;(\mathbf{S}^+\mathbf{S}^+) &= \frac{M_+\;(\mathbf{S}^+\mathbf{S}^+) + M_-\;(\mathbf{S}^-\;\mathbf{S}^+)}{S_0^-} \\ N_+\;S_0^- + M\;(\mathbf{S}^-\mathbf{S}^-) &= \frac{N_+\;(\mathbf{S}^+\mathbf{S}^-) + N_-\;(\mathbf{S}^-\;\mathbf{S}^-)}{S_0^+}\end{aligned} \tag{198}$$

In the first four equations one can see redundant ones; indeed from them it follows

$$M_+ = N\,\frac{(\mathbf{S}^+\mathbf{S}^-)}{S_0^+}\,, \qquad N\,S_0^- = M_+\,,$$
$$N_- = M\,\frac{(\mathbf{S}^-\mathbf{S}^+)}{S_0^-}\,, \qquad M\,S_0^+ = N_-\,,$$

they both result in

$$(\mathbf{S}^+\mathbf{S}^-) = S_0^+ S_0^- \qquad \Longleftrightarrow \qquad S_0 S_0 - \mathbf{S}\mathbf{S} = S_0' S_0' - \mathbf{S}'\mathbf{S}' = \text{inv}\,.$$

Therefore, the system (198) is equivalent to

$$\begin{aligned} M \;&= \frac{N_-}{S_0^+}\,, \qquad N = \frac{M_+}{S_0^-}\,, \\ (S_0^+ S_0^-)\;M_- + (\mathbf{S}^+\mathbf{S}^+)\;M_+ &= (\mathbf{S}^+\mathbf{S}^+)\;M_+ + (\mathbf{S}^+\;\mathbf{S}^-)\;M_- \\ (S_0^+ S_0^-)\;N_+ + (\mathbf{S}^-\mathbf{S}^-)\;N_- &= (\mathbf{S}^+\mathbf{S}^-)\;N_+ \;+ (\mathbf{S}^-\;\mathbf{S}^-)\;N_-\,; \end{aligned} \tag{199}$$

two last equations give

$$[\;(S_0^+ S_0^-) - (\mathbf{S}^+\;\mathbf{S}^-)\;]\;M_- = 0\,, \qquad [\;(S_0^+ S_0^-) - (\mathbf{S}^+\mathbf{S}^-)\;]\;N_+ = 0\,,$$

or

$$\begin{aligned} [\;(S_0 S_0 - \mathbf{S}\mathbf{S}) - (S_0' S_0' - \mathbf{S}'\mathbf{S}')\;]\;M_-\,, \qquad & M_- \qquad \text{is arbitrary}\,, \\ [\;(S_0 S_0 - \mathbf{S}\mathbf{S}) - (S_0' S_0' - \mathbf{S}'\mathbf{S}')\;]\;\;N_+ = 0\,, \qquad & N_+ \qquad \text{is arbitrary}\,. \end{aligned} \tag{200}$$

Thus, solution for transitivity problem is given by

$$\begin{aligned} \mathbf{n} &= N_+\;\mathbf{S}^+ + N_-\;\mathbf{S}^- + \frac{M_+}{S_0^-}\;(\mathbf{S}^- \times \mathbf{S}^+)\,, \\ \mathbf{m} &= M_+\;\mathbf{S}^+ + M_-\;\mathbf{S}^- + \frac{N_-}{S_0^+}\;(\mathbf{S}^- \times \mathbf{S}^+)\,, \\ n_0 &= \frac{M_+\;(\mathbf{S}^+\mathbf{S}^+) + M_-\;(\mathbf{S}^-\mathbf{S}^+)}{S_0^-}\,, \\ m_0 &= -\frac{N_+\;(\mathbf{S}^+\mathbf{S}^-) + N_-\;(\mathbf{S}^-\mathbf{S}^-)}{S_0^+}\;; \end{aligned} \tag{201}$$

additional restrictions on 4 parameters $M_\pm, N_\pm$ must hold:

$$n_0^2 + \mathbf{n}^2 - m_0^2 - \mathbf{m}^2 = 1\,, \qquad n_0 m_0 + \mathbf{n}\mathbf{m} = 0\,;$$

These solutions may include many different cases. For example, let us consider a simpler case when $M_+ = 0$ and $N_- = 0$:

$$\begin{aligned} n_0 &= \frac{M_-\;(\mathbf{S}^+\mathbf{S}^-)}{S_0^-} = M_- S_0^+\,, \qquad \mathbf{n} = N_+\;\mathbf{S}^+\,, \\ m_0 &= -\frac{N_+\;(\mathbf{S}^+\mathbf{S}^-)}{S_0^+} = -N_+ S_0^-\;\,, \qquad \mathbf{m} = M_-\;\mathbf{S}^-\,, \end{aligned} \tag{202}$$

and two restrictions must hold

$$n_0 m_0 + \mathbf{nm} = 0 \; , \qquad n_0^2 + \mathbf{n}^2 - m_0^2 - \mathbf{m}^2 = 1 \; .$$

The first holds identically

$$n_0 m_0 + \mathbf{nm} = -\frac{M_- \; (\mathbf{S}^+\mathbf{S}^-)}{S_0^-} \; \frac{N_+ \; (\mathbf{S}^+\mathbf{S}^-)}{S_0^+} + N_+ \; \mathbf{S}^+ \; M_- \; \mathbf{S}^- =$$
$$= M_- N_+ \;\; (\mathbf{S}^+\mathbf{S}^-) \;\; \frac{[\; S_0^+ S_0^- - (\mathbf{S}^+\mathbf{S}^-) \;]}{S_0^+ S_0^-} \equiv 0 \; .$$

The second one gives restriction on M_-^2 and N_+^2:

$$M_-^2 \; (S_0^+ S_0^+ - \mathbf{S}^-\mathbf{S}^-) - N_+^2 \; (S_0^- S_0^- - \mathbf{S}^+\mathbf{S}^+) = 1 \; ,$$

where

$$S_0^+ S_0^+ - \mathbf{S}^-\mathbf{S}^- = (S_0^2 - \mathbf{S}^2) + (S_0^{'2} - \mathbf{S}^{'2}) + 2(S_0 S_0' + \mathbf{SS}') \; ,$$
$$S_0^- S_0^- - \mathbf{S}^+\mathbf{S}^+ = (S_0^2 - \mathbf{S}^2) + (S_0^{'2} - \mathbf{S}^{'2}) - 2(S_0 S_0' + \mathbf{SS}') \; .$$

One may reach more simplicity in (202) by two ways:

$\underline{N_+ = 0}$ (pure Lorentz boost)

$$\begin{array}{c} n_0 = M_- S_0^+ \; , \qquad \mathbf{n} = 0 \; , \\ m_0 = 0 \;\; , \qquad \mathbf{m} = M_- \; \mathbf{S}^- \; , \\ k_0 = M_- S_0^+ \; , \qquad \mathbf{k} = M_- \mathbf{S}^- \; , \\ M_-^2 \; (S_0^+ S_0^+ - \mathbf{S}^-\mathbf{S}^-) = 1 \; ; \end{array} \tag{203}$$

the simplest example of that type is

$$\begin{array}{c} (S_0, S_1, S_2, S_3) \qquad \Longrightarrow \qquad (S_0', S_1, S_2, S_3') \; , \\ k_0 = M_-(S_0 + S_0') \; , \qquad k_1 = 0, \; k_2 = 0, \; k_3 = M_-(S_3 - S_3') \; , \\ M_-^2 \;\; = \dfrac{1}{[(S_0 - S_3') + (S_0' + S_3')] \; [(S_0 + S_3') + (S_0' - S_3')]} \; . \end{array} \tag{204}$$

$\underline{M_- = 0}$ (combination of boost and rotation)

$$\begin{array}{c} n_0 = 0 \; , \qquad \mathbf{n} = N_+ \; \mathbf{S}^+ \; , \\ m_0 = -N_+ S_0^- \;\; , \qquad \mathbf{m} = 0 \; , \\ k_0 = -i \; N_+ S_0^- \; , \qquad \mathbf{k} = -i \; N_+ \mathbf{S}^+ \; , \\ -N_+^2 \; (S_0^- S_0^- - \mathbf{S}^+\mathbf{S}^+) = 1 \; ; \end{array} \tag{205}$$

the simplest example of that type is

$$\begin{array}{c} (S_0, S_1, S_2, S_3) \qquad \Longrightarrow \qquad (S_0', -S_1, -S_2, S_3') \; , \\ k_0 = N_+(S_0 - S_0') \; , \qquad k_1 = 0, \; k_2 = 0, \; k_3 = N_+(S_3 + S_3') \; , \\ N_+^2 \;\; = \dfrac{-1}{(S_0 - S_0')^2 - (S_3' + S_3')^2} \; . \end{array}$$

It is easy to show that $B_+^2 > 0$:

$$
(S_0, S_3) = J(\text{ch}\,\gamma,\ \text{sh}\,\gamma)\ , \qquad (S_0', S_3') = J(\text{ch}\,\gamma',\ \text{sh}\,\gamma')\ ,
$$
$$
N_+^2 \;=\; \frac{-1}{2J^2[\,1 - (\text{ch}\,\gamma\ \text{ch}\,\gamma' + (\text{sh}\,\gamma\ \text{sh}\,\gamma')]} = \frac{1}{2J^2\ [\ \text{ch}\,(\gamma' - \gamma) - 1\ ]} > 0\ . \tag{206}
$$

Let us verify the formulas (203) (at real k_a) when turning to eqs. (185): they take the form

$$
\left|\begin{array}{rrrr} k_0 & -k_1 & -k_2 & -k_3 \\ -k_1 & k_0 & ik_3 & -ik_2 \\ -k_2 & -ik_3 & k_0 & ik_1 \\ -k_3 & ik_2 & -ik_1 & k_0 \end{array}\right| \left|\begin{array}{c} S_0 \\ S_1 \\ S_2 \\ S_3 \end{array}\right| = \left|\begin{array}{rrrr} k_0 & k_1 & k_2 & k_3 \\ k_1 & k_0 & ik_3 & -ik_2 \\ k_2 & -ik_3 & k_0 & ik_1 \\ k_3 & ik_2 & -ik_1 & k_0 \end{array}\right| \left|\begin{array}{c} S_0' \\ S_1' \\ S_2' \\ S_3' \end{array}\right| ,
$$

$$
\left|\begin{array}{rrrr} k_0 & -k_1 & -k_2 & -k_3 \\ -k_1 & k_0 & -ik_3 & ik_2 \\ -k_2 & ik_3 & k_0 & -ik_1 \\ -k_3 & -ik_2 & ik_1 & k_0 \end{array}\right| \left|\begin{array}{c} S_0 \\ S_1 \\ S_2 \\ S_3 \end{array}\right| = \left|\begin{array}{rrrr} k_0 & k_1 & k_2 & k_3 \\ k_1 & k_0 & -ik_3 & ik_2 \\ k_2 & ik_3 & k_0 & -ik_1 \\ k_3 & -ik_2 & ik_1 & k_0 \end{array}\right| \left|\begin{array}{c} S_0' \\ S_1' \\ S_2' \\ S_3' \end{array}\right| . \tag{207}
$$

It suffices to check only the first system that looks

$$
\begin{aligned}
k_0S_0 - k_1S_1 - k_2S_2 - k_3S_3 &= k_0S_0' + k_1S_1' + k_2S_2' + k_3S_3'\ , \\
-k_1S_0 + k_0S_1 + ik_3S_2 - ik_2S_3 &= k_1S_0' + k_0S_1' + ik_3S_2' - ik_2S_3'\ , \\
-k_2S_0 - ik_3S_1 + k_0S_2 + ik_1S_3 &= k_2S_0' - ik_3S_1' + k_0S_2' + ik_1S_3'\ , \\
-k_3S_0 + ik_2S_1 - ik_1S_2 + k_0S_3 &= k_3S_0' + ik_2S_1' - ik_1S_2' + k_0S_3'\ ,
\end{aligned}
$$

or

$$
\begin{aligned}
k_0S_0^- - \mathbf{k}\mathbf{S}^+ &= 0\ , \\
-k_1S_0^+ + k_0S_1^- + ik_3S_2^- - ik_2S_3^- &= 0\ , \\
-k_2S_0^+ - ik_3S_1^- + k_0S_2^- + ik_1S_3^- &= 0\ , \\
-k_3S_0^+ + ik_2S_1^- - ik_1S_2^- + k_0S_3^- &= 0\ ,
\end{aligned}
$$

or

$$
k_0S_0^- - \mathbf{k}\mathbf{S}^+ = 0\ , \qquad -\mathbf{k}S_0^+ + k_0\mathbf{S}^- + i\ \mathbf{S}^- \times \mathbf{k} = 0\ ;
$$

the later give identities when allowing for eqs. (203):

$$
M_-S_0^+S_0^- - M_-\mathbf{S}^-\mathbf{S}^+ = 0\ , \qquad -M_-\mathbf{S}^-S_0^+ + M_-S_0^+\mathbf{S}^- + i\ \mathbf{S}^- \times M_-\mathbf{S}^- = 0\ .
$$

Similarly, let us verify the formulas (205) (at imaginary k_a): eqs. (185) take the form

$$
\left|\begin{array}{rrrr} -k_0 & k_1 & k_2 & k_3 \\ k_1 & -k_0 & -ik_3 & ik_2 \\ k_2 & ik_3 & -k_0 & -ik_1 \\ k_3 & -ik_2 & ik_1 & -k_0 \end{array}\right| \left|\begin{array}{c} S_0 \\ S_1 \\ S_2 \\ S_3 \end{array}\right| = \left|\begin{array}{rrrr} k_0 & k_1 & k_2 & k_3 \\ k_1 & k_0 & ik_3 & -ik_2 \\ k_2 & -ik_3 & k_0 & ik_1 \\ k_3 & ik_2 & -ik_1 & k_0 \end{array}\right| \left|\begin{array}{c} S_0' \\ S_1' \\ S_2' \\ S_3' \end{array}\right| ,
$$

$$
\left| \begin{array}{rrrr} k_0 & -k_1 & -k_2 & -k_3 \\ -k_1 & k_0 & -ik_3 & ik_2 \\ -k_2 & ik_3 & k_0 & -ik_1 \\ -k_3 & -ik_2 & ik_1 & k_0 \end{array} \right| \left| \begin{array}{c} S_0 \\ S_1 \\ S_2 \\ S_3 \end{array} \right| = \left| \begin{array}{rrrr} -k_0 & -k_1 & -k_2 & -k_3 \\ -k_1 & -k_0 & ik_3 & -ik_2 \\ -k_2 & -ik_3 & -k_0 & ik_1 \\ -k_3 & ik_2 & -ik_1 & -k_0 \end{array} \right| \left| \begin{array}{c} S'_0 \\ S'_1 \\ S'_2 \\ S'_3 \end{array} \right| .
$$

It suffices to check the first equation that gives

$$
\begin{aligned}
- k_0 S_0 + k_1 S_1 + k_2 S_2 + k_3 S_3 &= k_0 S'_0 + k_1 S'_1 + k_2 S'_2 + k_3 S'_3 \;, \\
k_1 S_0 - k_0 S_1 - ik_3 S_2 + ik_2 S_3 &= k_1 S'_0 + k_0 S'_1 + ik_3 S'_2 - ik_2 S'_3 \;, \\
k_2 S_0 + ik_3 S_1 - k_0 S_2 - ik_1 S_3 &= k_2 S'_0 - ik_3 S'_1 + k_0 S'_2 + ik_1 S'_3 \;, \\
k_3 S_0 - ik_2 S_1 + ik_1 S_2 - k_0 S_3 &= k_3 S'_0 + ik_2 S'_1 - ik_1 S'_2 + k_0 S'_3 \;,
\end{aligned}
$$

or

$$
\begin{aligned}
- k_0 S_0^+ + k_1 S_1^- + k_2 S_2^- + k_3 S_3^- &= 0 \;, \\
k_1 S_0^- - k_0 S_1^+ - ik_3 S_2^+ + ik_2 S_3^+ &= 0 \;, \\
k_2 S_0^- + ik_3 S_1^+ - k_0 S_2^+ - ik_1 S_3^+ &= 0 \;, \\
k_3 S_0^- - ik_2 S_1^+ + ik_1 S_2^+ - k_0 S_3^+ &= 0 \;,
\end{aligned}
$$

that is

$$
- k_0 S_0^+ + \mathbf{k} \mathbf{S}^- = 0 \;, \qquad \mathbf{k} S_0^- - k_0 \mathbf{S}^+ - i \mathbf{S}^+ \times \mathbf{k} = 0 \;;
$$

the later give identities when allowing for eqs. (205):

$$
i\, N_+ S_0^- S_0^+ - i\, N_+ \mathbf{S}^+ \mathbf{S}^- = 0 \;, \qquad -i\, N_+ \mathbf{S}^+ S_0^- + i\, N_+ S_0^- \mathbf{S}^+ + i \mathbf{S}^+ \times i\, N_+ \mathbf{S}^+ = 0 \;.
$$

In the end of the Section let us turn again to relations providing solutions solution for transitivity problem (202):

$$
\begin{aligned}
\mathbf{n} &= N_+ \; \mathbf{S}^+ + N_- \; \mathbf{S}^- + \frac{M_+}{S_0^-} \, (\mathbf{S}^- \times \mathbf{S}^+) \;, \\
\mathbf{m} &= M_+ \; \mathbf{S}^+ + M_- \; \mathbf{S}^- + \frac{N_-}{S_0^+} \, (\mathbf{S}^- \times \mathbf{S}^+) \;, \\
n_0 &= \frac{M_+ \, (\mathbf{S}^+ \mathbf{S}^+) + M_- \, (\mathbf{S}^- \mathbf{S}^+)}{S_0^-} \;, \\
m_0 &= - \frac{N_+ \, (\mathbf{S}^+ \mathbf{S}^-) + N_- \, (\mathbf{S}^- \mathbf{S}^-) \;,}{S_0^+} \;\; ; \\
n_0 m_0 + \mathbf{n}\mathbf{m} = 0 \;, &\qquad n_0^2 + \mathbf{n}^2 - m_0^2 - \mathbf{m}^2 = 1 \;, \qquad ;
\end{aligned} \tag{208}
$$

and consider explicit form of restrictions (208) on 4 parameters $M_\pm, N_\pm$.

Allowing for relations

$$
n_0 m_0 = -(\; \frac{M_+}{S_0^-} \;\; \mathbf{S}^+ \mathbf{S}^+ + M_- S_0^+ \;) \;\; (\; \frac{N_-}{S_0^+} \;\; \mathbf{S}^- \mathbf{S}^- + N_+ S_0^- \;) =
$$

$$
= - \left[\; \frac{M_+ N_-}{S_0^+ S_0^-} \, (\mathbf{S}^+ \mathbf{S}^+)(\mathbf{S}^- \mathbf{S}^-) + M_+ N_+ \;\; \mathbf{S}^+ \mathbf{S}^+ + M_- N_- \;\; \mathbf{S}^- \mathbf{S}^- \;\; + M_- N_+ S_0^+ S_0^- \; \right]
$$

$$\mathbf{mn} = \Big(M_+ \, \mathbf{S}^+ + M_- \, \mathbf{S}^- + \frac{N_-}{S_0^+} \, (\mathbf{S}^- \times \mathbf{S}^+) \Big) \ \ \Big(N_+ \, \mathbf{S}^+ + N_- \, \mathbf{S}^- + \frac{M_+}{S_0^-} \, (\mathbf{S}^- \times \mathbf{S}^+) \Big) =$$

$$= M_+ N_+ (\mathbf{S}^+\mathbf{S}^+) + M_- N_- \, (\mathbf{S}^-\mathbf{S}^-) + (M_+ N_- + M_- N_+)(\mathbf{S}^+ \, \mathbf{S}^-) +$$

$$+ \frac{N_- M_+}{S_0^+ S_0^-} \, [\, (\mathbf{S}^-\mathbf{S}^-)(\mathbf{S}^+\mathbf{S}^+) - (\mathbf{S}^+\mathbf{S}^-)^2 \,] =$$

$$= M_+ N_+ (\mathbf{S}^+\mathbf{S}^+) + M_- N_- \, (\mathbf{S}^-\mathbf{S}^-) + M_- N_+ S_0^+ S_0^- + \frac{N_- M_+}{S_0^+ S_0^-} \ \ (\mathbf{S}^-\mathbf{S}^-)(\mathbf{S}^+\mathbf{S}^+)$$

we arrive at identity $0 = 0$:

$$n_0 m_0 + \mathbf{nm} =$$

$$= - \frac{M_+ N_-}{S_0^+ S_0^-} \, (\mathbf{S}^+\mathbf{S}^+)(\mathbf{S}^-\mathbf{S}^-) - M_+ N_+ \ \ \mathbf{S}^+\mathbf{S}^+ - M_- N_- \ \ \mathbf{S}^-\mathbf{S}^- \ - M_- N_+ S_0^+ S_0^- +$$

$$+ M_+ N_+ (\mathbf{S}^+\mathbf{S}^+) + M_- N_- \, (\mathbf{S}^-\mathbf{S}^-) + M_- N_+ S_0^+ S_0^- + \frac{N_- M_+}{S_0^+ S_0^-} \ \ (\mathbf{S}^-\mathbf{S}^-)(\mathbf{S}^+\mathbf{S}^+) =$$

$$= - M_- N_+ (S_0^+ S_0^- - S_0^+ S_0^-) \equiv 0 \, .$$

Now let us turn to second equation; allowing for relations

$$n_0^2 = \frac{M_+^2}{S_0^- S_0^-} \, (\mathbf{S}^+\mathbf{S}^+)^2 + 2 M_+ M_- \frac{S_0^+}{S_0^-} \, (\mathbf{S}^+\mathbf{S}^+) + M_-^2 \ S_0^+ S_0^+$$

$$= m_0^2 = \frac{N_-^2}{S_0^+ S_0^+} \, (\mathbf{S}^-\mathbf{S}^-)^2 + 2 N_+ N_- \frac{S_0^-}{S_0^+} \, (\mathbf{S}^-\mathbf{S}^-) + N_+^2 \ S_0^- S_0^- \ ,$$

$$\mathbf{n}^2 = (N_+ \, \mathbf{S}^+ + N_- \, \mathbf{S}^- + \frac{M_+}{S_0^-} \, (\mathbf{S}^- \times \mathbf{S}^+)) \ \ \ (N_+ \, \mathbf{S}^+ + N_- \, \mathbf{S}^- + \frac{M_+}{S_0^-} \, (\mathbf{S}^- \times \mathbf{S}^+)) =$$

$$= N_+^2 (\mathbf{S}^+\mathbf{S}^+) + N_-^2 (\mathbf{S}^-\mathbf{S}^-) + 2 N_+ N_- \ S_0^+ S_0^- - S_0^+ S_0^+ M_+^2 + \frac{M_+^2}{S_0^- S_0^-} \, (\mathbf{S}^+\mathbf{S}^+)(\mathbf{S}^-\mathbf{S}^-) \ ,$$

$$\mathbf{m}^2 = (M_+ \, \mathbf{S}^+ + M_- \, \mathbf{S}^- + \frac{N_-}{S_0^+} \, (\mathbf{S}^- \times \mathbf{S}^+)) \ (M_+ \, \mathbf{S}^+ + M_- \, \mathbf{S}^- + \frac{N_-}{S_0^+} \, (\mathbf{S}^- \times \mathbf{S}^+)) =$$

$$= M_+^2 (\mathbf{S}^+\mathbf{S}^+) + M_-^2 (\mathbf{S}^-\mathbf{S}^-) + 2 M_+ M_- \ S_0^+ S_0^- - S_0^- S_0^- N_-^2 + \frac{N_-^2}{S_0^+ S_0^+} \, (\mathbf{S}^+\mathbf{S}^+)(\mathbf{S}^-\mathbf{S}^-) \ ,$$

we reduce the second equation to the form

$$\frac{M_+^2}{S_0^- S_0^-} \, (\mathbf{S}^+\mathbf{S}^+)^2 + 2 M_+ M_- \frac{S_0^+}{S_0^-} \, (\mathbf{S}^+\mathbf{S}^+) + M_-^2 \ S_0^+ S_0^+ +$$

$$+ N_+^2 (\mathbf{S}^+\mathbf{S}^+) + N_-^2 (\mathbf{S}^-\mathbf{S}^-) + 2 N_+ N_- \ S_0^+ S_0^- - S_0^+ S_0^+ M_+^2 + \frac{M_+^2}{S_0^- S_0^-} \, (\mathbf{S}^+\mathbf{S}^+)(\mathbf{S}^-\mathbf{S}^-) -$$

$$- \frac{N_-^2}{S_0^+ S_0^+} \, (\mathbf{S}^-\mathbf{S}^-)^2 - 2 N_+ N_- \frac{S_0^-}{S_0^+} \, (\mathbf{S}^-\mathbf{S}^-) - N_+^2 \ S_0^- S_0^- -$$

$$- M_+^2 (\mathbf{S}^+\mathbf{S}^+) - M_-^2 (\mathbf{S}^-\mathbf{S}^-) - 2 M_+ M_- \ S_0^+ S_0^- + S_0^- S_0^- N_-^2 - \frac{N_-^2}{S_0^+ S_0^+} \, (\mathbf{S}^+\mathbf{S}^+)(\mathbf{S}^-\mathbf{S}^-) = 1,$$

which after evident simplification reads

$$M_+^2 \left[\frac{1}{S_0^- S_0^-} (\mathbf{S}^+\mathbf{S}^+)^2 - S_0^+ S_0^+ + \frac{1}{S_0^- S_0^-} (\mathbf{S}^+\mathbf{S}^+)(\mathbf{S}^-\mathbf{S}^-) - (\mathbf{S}^+\mathbf{S}^+) \right]$$
$$+2M_+M_- \left(\frac{S_0^+}{S_0^-} (\mathbf{S}^+\mathbf{S}^+) - S_0^+ S_0^- \right) + M_-^2 \ \left[\ S_0^+ S_0^+ - (\mathbf{S}^-\mathbf{S}^-) \ \right] =$$
$$= N_-^2 \left[\frac{1}{S_0^+ S_0^+} (\mathbf{S}^-\mathbf{S}^-)^2 - S_0^- S_0^- + \frac{1}{S_0^+ S_0^+} (\mathbf{S}^+\mathbf{S}^+)(\mathbf{S}^-\mathbf{S}^-) - (\mathbf{S}^-\mathbf{S}^-) \right] +$$
$$+2N_+N_- \left(\frac{S_0^-}{S_0^+} (\mathbf{S}^-\mathbf{S}^-) - S_0^+ S_0^- \right) + N_+^2 \ \left[\ S_0^- S_0^- - (\mathbf{S}^+\mathbf{S}^+) \ \right] + 1 \ . \tag{209}$$

This equation on 4 parameter let us designate as

$$F(S, S';\ M_\pm, N_\pm) = 1$$

This second-order surface in 4-dimensional real space (equation on 4 parameters $M_\pm, N_\pm$, coefficients of which depend on 4 parameters $S_0^+ = a,\ \mathbf{S}^+\mathbf{S}^+,\ S_0^- = b,\ \mathbf{S}^-\mathbf{S}^-$) describes a 3-dimensional manyfold of all possible transitive transformations L relating two Stokes 4-vectors: $LS = S'$.

In the context of polarization optics this means that at given Stokes vector S_a and S'_a a Mueller matrix referring them cannot be measured unambiguously – the freedom is given by 3-dimensional manifold of corresponding Mueller matrices.

To determine (or to measure) a Myuller matrix of the optical device in fact one need 4 pairs of Stokes vectors (or 4 different experiments with light):

$$\begin{array}{ll}
L_1 S_{(1)} = S'_{(1)};, & F(M_\pm, N_\pm;\ \ S_{(1)}, S'_{(1)}) = 1 \ ; \\
L_2 S_{(2)} = S'_{(2)};, & F(M_\pm, N_\pm;\ \ S_{(2)}, S'_{(2)}) = 1 \ ; \\
L_3 S_{(3)} = S'_{(3)};, & F(M_\pm, N_\pm;\ \ S_{(3)}, S'_{(3)}) = 1 \ ; \\
L_4 S_{(4)} = S'_{(4)};, & F(M_\pm, N_\pm;\ \ S_{(4)}, S'_{(4)}) = 1 \ .
\end{array} \tag{210}$$

11 Polar decomposition of Mueller (Lorentz) matrices

Because Euclidean rotation and Lorentzian significantly differ physically in the context of Mueller matrices let briefly consider the problem of factorization arbitrary Lorentz transformation into product of rotation and boost (surely this matter is well known in the literature on relativistic kinematics, see for instance in [4]).

It is convenient to make analysis upon spinor representations of the Lorentz group:

$$B(k) = k_0 + \mathbf{k}\,\vec{\sigma} = (a_0 - i\mathbf{a}\,\vec{\sigma})(b_0 + \mathbf{b}\,\vec{\sigma})$$
$$= (a_0 b_0 - i\ \mathbf{a}\ \mathbf{b}) + (\ a_0\ \mathbf{b} + \mathbf{a} \times \mathbf{b} - i\ b_0\ \mathbf{a}\ )\ \vec{\sigma}\ ,$$
$$k_0^* + \mathbf{k}^*\vec{\sigma} = (a_0 + i\mathbf{a}\vec{\sigma})(b_0 + \mathbf{b}\vec{\sigma}) =$$
$$= (a_0 b_0 + i\ \mathbf{a}\ \mathbf{b}) + (\ a_0\ \mathbf{b} + \mathbf{a} \times \mathbf{b} + i\ b_0\ \mathbf{a}\ )\ \vec{\sigma}\ ; \tag{211}$$

which is equivalent to the system

$$k_0 = (a_0 b_0 - i\, \mathbf{a}\, \mathbf{b})\,, \qquad k_0^* = (a_0 b_0 + i\, \mathbf{a}\,, \mathbf{b})$$
$$\mathbf{k} = (\, a_0\, \mathbf{b} + \mathbf{a} \times \mathbf{b} - i\, b_0\, \mathbf{a}\,)\,, \qquad \mathbf{k}^* = (\, a_0\, \mathbf{b} + \mathbf{a} \times \mathbf{b} + i\, b_0\, \mathbf{a}\,)$$

or

$$\frac{k_0 + k_0^*}{2} = a_0 b_0\,, \qquad \frac{k_0 - k_0^*}{2i} = -\, \mathbf{a}\, \mathbf{b}\,,$$
$$\frac{\mathbf{k} + \mathbf{k}^*}{2} = \ (\, a_0\, \mathbf{b} + \mathbf{a} \times \mathbf{b})\,, \qquad \frac{i\mathbf{k} - i\mathbf{k}^*}{2} = b_0\, \mathbf{a}\,. \tag{212}$$

With additional restrictions:

$$a_0^2 + \mathbf{a}^2 = +1\,, \qquad a_0 = \pm\, \sqrt{1 - \mathbf{a}^2}\,,$$
$$b_0^2 - \mathbf{b}^2 = +1, \qquad b_0 = +\sqrt{1 + \mathbf{b}^2} \geq +1 \tag{213}$$

eqs. (212) take the form

$$n_0 = \pm\, \sqrt{1 - \mathbf{a}^2}\ \ \sqrt{1 + \mathbf{b}^2}\,, \qquad m_0 = -\, \mathbf{a}\, \mathbf{b}\,,$$
$$\mathbf{m} = \ (\, \pm\, \sqrt{1 - \mathbf{a}^2}\, \mathbf{b} + \mathbf{a} \times \mathbf{b})\,, \qquad \mathbf{n} = \sqrt{1 + \mathbf{b}^2}\ \ \mathbf{a}\,. \tag{214}$$

From whence it follows

$$n_0 = \pm\, \sqrt{1 - \mathbf{a}^2}\ \ \sqrt{1 + \mathbf{b}^2}\,, \qquad m_0 = -\, \mathbf{a}\, \mathbf{b}\,,$$
$$\frac{\mathbf{m}}{n_0} = \frac{\mathbf{b}}{\sqrt{1 + \mathbf{b}^2}} + \frac{\mathbf{a}}{\pm\sqrt{1 - \mathbf{a}^2}} \times \frac{\mathbf{b}}{\sqrt{1 + \mathbf{b}^2}}\,, \qquad \frac{\mathbf{n}}{n_0} = \frac{\mathbf{a}}{\pm\sqrt{1 - \mathbf{a}^2}}\,. \tag{215}$$

With the help of variables $\mathbf{A}, \mathbf{B}$:

$$\frac{\mathbf{b}}{\sqrt{1 + \mathbf{b}^2}} = \mathbf{B}\,, \ \ b_0 = \sqrt{1 + \mathbf{b}^2} = \frac{1}{\sqrt{1 - \mathbf{B}^2}}\,, \ \ \mathbf{b} = \frac{\mathbf{B}}{\sqrt{1 - \mathbf{B}^2}}\,,$$
$$\frac{\mathbf{a}}{\pm\sqrt{1 - \mathbf{a}^2}} = \pm\mathbf{A}\,, \ \ a_0 = \pm\sqrt{1 - \mathbf{a}^2} = \frac{1}{\pm\sqrt{1 + \mathbf{A}^2}}\,, \ \ \mathbf{a} = \frac{\mathbf{A}}{\sqrt{1 + \mathbf{A}^2}}\,;$$

eqs. (214) read

$$n_0 = \pm\, \frac{1}{\sqrt{1 + \mathbf{A}^2}}\, \frac{1}{\sqrt{1 - \mathbf{B}^2}}\,, \qquad m_0 = -\, \frac{\mathbf{A}}{\sqrt{1 + \mathbf{A}^2}}\, \frac{\mathbf{B}}{\sqrt{1 - \mathbf{B}^2}}\,,$$
$$\frac{\mathbf{m}}{n_0} = \mathbf{B} + \mathbf{A} \times \mathbf{B}\,, \qquad \frac{\mathbf{n}}{n_0} = \mathbf{A}\,. \tag{216}$$

The vector $\mathbf{B}$ may be resolved into a linear combination

$$\mathbf{B} = \nu\, \mathbf{n} + \mu\, \mathbf{m} + \sigma\, \mathbf{n} \times \mathbf{m}$$

which must obey

$$\frac{\mathbf{m}}{n_0} = \nu\, \mathbf{n} + \mu\, \mathbf{m} + \sigma\, \mathbf{n} \times \mathbf{m} + \frac{\mathbf{n}}{n_0} \times (\nu\, \mathbf{n} + \mu\, \mathbf{m} + \sigma\, \mathbf{n} \times \mathbf{m})$$

or

$$\frac{\mathbf{m}}{n_0} = \nu\, \mathbf{n} + \mu\, \mathbf{m} + \sigma\, \mathbf{n} \times \mathbf{m} + \frac{\mu}{n_0}\, \mathbf{n} \times \mathbf{m} + \frac{\sigma}{n_0} (\mathbf{nm})\, \mathbf{n} - \frac{\sigma}{n_0}\, (\mathbf{n}^2)\, \mathbf{m} \; .$$

Therefore, we have the system

$$\nu + \frac{\sigma}{n_0} (\mathbf{nm}) = 0 \; , \qquad \frac{1}{n_0} = \mu - \frac{\sigma\, (\mathbf{n}^2)}{n_0} \; , \qquad \sigma + \frac{\mu}{n_0} = 0$$

with evident solution

$$\sigma = -\frac{1}{n_0^2 + \mathbf{n}^2} \; , \quad \mu = \frac{n_0}{n_0^2 + \mathbf{n}^2} \; , \quad \nu = \frac{(\mathbf{nm})}{n_0}\, \frac{1}{n_0^2 + \mathbf{n}^2} = -m_0\, \frac{1}{n_0^2 + \mathbf{n}^2} \; .$$

Thus, the factorization we need is found:

$$k_0 + \mathbf{k}\, \vec{\sigma} = (\frac{n_0 - i\mathbf{n}\, \vec{\sigma}}{\sqrt{n_0^2 + \mathbf{n}^2}})\, (\frac{1}{\sqrt{1 - \mathbf{B}^2}} + \frac{\mathbf{B}\, \vec{\sigma}}{\sqrt{1 - \mathbf{B}^2}}) \; ,$$

$$\mathbf{B} = \frac{n_0\, \mathbf{m} - m_0\, \mathbf{n} + \; \mathbf{m} \times \mathbf{n}}{n_0^2 + \mathbf{n}^2} \; . \qquad (217)$$

The problem of factorization may be solved easily with opposite order:

$$k_0 + \mathbf{k}\, \vec{\sigma} = (b_0 + \mathbf{b}\, \vec{\sigma})(a_0 - i\mathbf{a}\, \vec{\sigma})$$
$$= (a_0 b_0 - i\, \mathbf{a}\, \mathbf{b}) + (\; a_0\, \mathbf{b} - \mathbf{a} \times \mathbf{b} - i\, b_0\, \mathbf{a}\;)\, \vec{\sigma} \; ,$$

$$k_0^* + \mathbf{k}^* \vec{\sigma} = (b_0 + \mathbf{b}\vec{\sigma})(a_0 + i\mathbf{a}\vec{\sigma}) =$$
$$= (a_0 b_0 + i\, \mathbf{a}\, \mathbf{b}) + (\; a_0\, \mathbf{b} - \mathbf{a} \times \mathbf{b} + i\, b_0\, \mathbf{a}\;)\, \vec{\sigma} \; ; \qquad (218)$$

it reduces to the system (in comparison with (211) only the sign at the vector product has been changed on opposite)

$$k_0 = (a_0 b_0 - i\, \mathbf{a}\, \mathbf{b}) \; , \qquad k_0^* = (a_0 b_0 + i\, \mathbf{a}\, \mathbf{b}) \; ,$$
$$\mathbf{k} = (\; a_0\, \mathbf{b} - \mathbf{a} \times \mathbf{b} - i\, b_0\, \mathbf{a}\;) \; , \qquad \mathbf{k}^* = (\; a_0\, \mathbf{b} - \mathbf{a} \times \mathbf{b} + i\, b_0\, \mathbf{a}\; , \;)$$

or

$$n_0 = a_0 b_0 \; , \qquad m_0 = -\; \mathbf{a}\, \mathbf{b} \; ,$$
$$\mathbf{m} = \; (\; a_0\, \mathbf{b} - \mathbf{a} \times \mathbf{b}) \; , \qquad \mathbf{n} = b_0\, \mathbf{a} \; . \qquad (219)$$

Further analysis is the same, the final result is

$$k_0 + \mathbf{k}\, \vec{\sigma} = (\frac{1}{\sqrt{1 - \mathbf{B}^2}} + \frac{\mathbf{B}\, \vec{\sigma}}{\sqrt{1 - \mathbf{B}^2}})\, (\frac{n_0 - i\mathbf{n}\, \vec{\sigma}}{\sqrt{n_0^2 + \mathbf{n}^2}}) \; ,$$

$$\mathbf{B} = \frac{n_0\, \mathbf{m} - m_0\, \mathbf{n} - \; \mathbf{m} \times \mathbf{n}}{n_0^2 + \mathbf{n}^2} \; . \qquad (220)$$

12 Superposition of two boost, Thomas precession in optics

Superposition of two boosts is given by

$$
\begin{aligned}
& (b'_0 + \mathbf{b}'\,\vec{\sigma})\,(b_0 + \mathbf{b}\,\vec{\sigma}) = \\
& = b'_0 b_0 + \mathbf{b}'\mathbf{b} + (b'_0 \mathbf{b} + b_0 \mathbf{b}' + i\mathbf{b}' \times \mathbf{b}\,)\,\vec{\sigma} = \\
& = (n_0 + i m_0) + (-i\mathbf{n} + \mathbf{m})\,\vec{\sigma}
\end{aligned}
$$

where

$$
n_0 = b'_0 b_0 + \mathbf{b}'\mathbf{b}\,, \qquad \mathbf{n} = \mathbf{b} \times \mathbf{b}'\,,
$$
$$
m_0 = 0\,, \qquad \mathbf{m} = b'_0 \mathbf{b} + b_0 \mathbf{b}'\,. \tag{221}
$$

And factorization we need to describe Thomas precession is (see (217))

$$
\begin{aligned}
& (b'_0 + \mathbf{b}'\,\vec{\sigma})\,(b_0 + \mathbf{b}\,\vec{\sigma}) = (n_0 + i m_0) + (-i\mathbf{n} + \mathbf{m})\,\vec{\sigma} = \\
& = \Big(\frac{n_0 - i\mathbf{n}\,\vec{\sigma}}{\sqrt{n_0^2 + \mathbf{n}^2}}\Big)\,\Big(\frac{1}{\sqrt{1-\mathbf{B}^2}} + \frac{\mathbf{B}\,\vec{\sigma}}{\sqrt{1-\mathbf{B}^2}}\Big)\,,
\end{aligned}
$$

where

$$
\begin{aligned}
& \mathbf{B} = \frac{n_0\,\mathbf{m} - m_0\,\mathbf{n} + \ \mathbf{m} \times \mathbf{n}}{n_0^2 + \mathbf{n}^2} = \\
& = \frac{(b'_0 b_0 + \mathbf{b}'\mathbf{b})\,(b'_0 \mathbf{b} + b_0 \mathbf{b}') + \ (b'_0 \mathbf{b} + b_0 \mathbf{b}') \times (\mathbf{b} \times \mathbf{b}')}{(b'_0 b_0 + \mathbf{b}'\mathbf{b})^2 + (\mathbf{b} \times \mathbf{b}')^2}
\end{aligned} \tag{222}
$$

13 Polarization of the light, Mueller formalism in isotropic basis

At working with Lorentz group, a special, so-called isotropic formalism by Newmann and Penrose [11], was elaborated. It can be used in the context of description of the light polarization as well. Let us relate this technique with the above treatment.

The Lorentz matrix (48), after the change of variables $A_0 = k_0$, $A_j = ik_j$, takes the form

$$
L = \hat{A}\hat{A}^* = \hat{A}^*\hat{A}\,, \quad \hat{A} = \left| \begin{array}{rrrr} k_0 & -k_1 & -k_2 & -k_3 \\ -k_1 & k_0 & -ik_3 & ik_2 \\ -k_2 & ik_3 & k_0 & -ik_1 \\ -k_3 & -ik_2 & ik_1 & k_0 \end{array} \right| , \ \hat{A}^* = \left| \begin{array}{rrrr} k_0^* & -k_1^* & -k_2^* & -k_3^* \\ -k_1^* & k_0^* & ik_3^* & -ik_2^* \\ -k_2^* & -ik_3^* & k_0^* & ik_1^* \\ -k_3^* & ik_2^* & -ik_1^* & k_0^* \end{array} \right| . \tag{223}
$$

Transforming to the isotropic basis is realized through

$$
L \Longrightarrow U = L_{isotr} = SLS^{-1}\,,
$$
$$
A \quad \Longrightarrow \quad A_{isotr} = SAS^{-1}\,, \qquad A^* \quad \Longrightarrow \quad L^*_{isotr} = SA^*S^{-1}\,, \tag{224}
$$

where

$$
S = \frac{1}{\sqrt{2}} \left| \begin{array}{rrrr} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -i & 0 \\ 0 & 1 & i & 0 \end{array} \right| , \qquad S^{-1} = \frac{1}{\sqrt{2}} \left| \begin{array}{rrrr} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & i & -i \\ 1 & -1 & 0 & 0 \end{array} \right| .
$$

For two factors we get

$$A_{isotr} = \left| \begin{array}{cccc} (k_0 - k_3) & 0 & -(k_1 + ik_2) & 0 \\ 0 & (k_0 + k_3) & 0 & -(k_1 - ik_2) \\ -(k_1 - ik_2) & 0 & (k_0 + k_3) & 0 \\ 0 & -(k_1 + ik_2) & 0 & (k_0 - k_3) \end{array} \right| ,$$

$$A^*_{isotr} = \left| \begin{array}{cccc} (k^*_0 - k^*_3) & 0 & 0 & -(k^*_1 - ik^*_2) \\ 0 & (k^*_0 + k^*_3) & -(k^*_1 + ik^*_2) & 0 \\ 0 & -(k^*_1 - ik^*_2) & (k^*_0 - k^*_3) & 0 \\ -(k^*_1 + ik^*_2) & 0 & 0 & (k^*_0 + k^*_3) \end{array} \right| . \qquad (225)$$

It is convenient to introduce the following notation:

$$B(k) = k_0 + \sigma^j k_j = \left| \begin{array}{cc} k_0 + k_3 & k_1 - ik_2 \\ k_1 + ik_2 & k_0 - k_3 \end{array} \right| = \left| \begin{array}{cc} a & d \\ c & b \end{array} \right| ,$$

$$B(\bar{k}^*) = \left| \begin{array}{cc} k^*_0 - k^*_3 & -(k^*_1 - ik^*_2) \\ -(k^*_1 + ik^*_2) & k^*_0 + k^*_3 \end{array} \right| = \left| \begin{array}{cc} b^* & -c^* \\ -d^* & a^* \end{array} \right| , \qquad (226)$$

then relations (225) read

$$A_{isotr} = \left| \begin{array}{rrrr} b & 0 & -c & 0 \\ 0 & a & 0 & -d \\ -d & 0 & a & 0 \\ 0 & -c & 0 & b \end{array} \right| , \qquad A^*_{isotr} = \left| \begin{array}{rrrr} b^* & 0 & 0 & -c^* \\ 0 & a^* & -d^* & 0 \\ 0 & -c^* & b^* & 0 \\ -d^* & 0 & 0 & a^* \end{array} \right| , \qquad (227)$$

and the Lorentz matrix in isotropic form looks as follows

$$U = L_{isotr} = \left| \begin{array}{rrrr} bb^* & cc^* & -cb^* & -bc^* \\ dd^* & aa^* & -ad^* & -da^* \\ -db^* & -ac^* & ab^* & dc^* \\ -bd^* & -ca^* & cd^* & ba^* \end{array} \right| . \qquad (228)$$

For boost transformations and Euclidean rotations we have respectively

$$B(k) = \text{ch}\ \frac{\beta}{2} + \text{sh}\ \frac{\beta}{2}\ \sigma^j e_j = \left| \begin{array}{cc} \text{ch}\ (\beta/2) + \text{sh}\ (\beta/2) e_3 & \text{sh}\ (\beta/2)(e_1 - ie_2) \\ \text{sh}\ (\beta/2)(e_1 + ie_2) & \text{ch}\ (\beta/2) - \text{sh}\ (\beta/2) e_3 \end{array} \right| ,$$

$$B(k) = \cos\ \frac{\phi}{2} - i \sin \frac{\phi}{2}\ \sigma^j e_j = \left| \begin{array}{cc} \cos\ (\phi/2) - i \sin\ (\beta/2) e_3 & -i \sin\ (\phi/2)(e_1 - ie_2) \\ -i \sin\ (\beta/2)(e_1 + ie_2) & \cos\ (\phi/2) + i \sin(\phi/2) e_3 \end{array} \right| . \qquad (229)$$

The matrix (228) acts in the space of the variables $Z = (Z_a)$, representing Stokes vector in isotropic basis:

$$Z' = UZ\ , \qquad Z = \frac{1}{\sqrt{2}} \left| \begin{array}{rrrr} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -i & 0 \\ 0 & 1 & i & 0 \end{array} \right| \left| \begin{array}{c} I \\ Ip_1 \\ Ip_2 \\ Ip_3 \end{array} \right| = \frac{1}{\sqrt{2}} \left| \begin{array}{r} I(1 + p_3) \\ I(1 - p_3) \\ I(p_1 - ip_2) \\ I(p_1 + ip_2) \end{array} \right| ,$$

in accordance with

$$\left| \begin{array}{r} I'(1+p'_3) \\ I'(1-p'_3) \\ I'(p'_1-ip'_2) \\ I'(p'_1+ip'_2) \end{array} \right| \left| \begin{array}{rrrr} bb^* & cc^* & -cb^* & -bc^* \\ dd^* & aa^* & -ad^* & -da^* \\ -db^* & -ac^* & ab^* & dc^* \\ -bd^* & -ca^* & cd^* & ba^* \end{array} \right| = \left| \begin{array}{r} I(1+p_3) \\ I(1-p_3) \\ I(p_1-ip_2) \\ I(p_1+ip_2) \end{array} \right| . \tag{230}$$

Let us specify six particular cases:

rotation $(0-1)$

$$U = \frac{1}{2} \left| \begin{array}{rrrr} (\text{ch}\ \beta+1) & (\text{ch}\ \beta-1) & -\text{sh}\ \beta & -\text{sh}\ \beta \\ (\text{ch}\ \beta-1) & (\text{ch}\ \beta+1) & -\text{sh}\ \beta & -\text{sh}\ \beta \\ -\text{sh}\ \beta & -\text{sh}\ \beta & (\text{ch}\ \beta+1) & (\text{ch}\ \beta-1) \\ -\text{sh}\ \beta & -\text{sh}\ \beta & (\text{ch}\beta-1) & (\text{ch}\ \beta+1) \end{array} \right| ,$$

rotation $(0-2)$

$$U = \frac{1}{2} \left| \begin{array}{rrrr} (\text{ch}\ \beta+1) & (\text{ch}\ \beta-1) & -i\ \text{sh}\ \beta & +i\ \text{sh}\ \beta \\ (\text{ch}\beta-1) & (\text{ch}\beta+1) & -i\ \text{sh}\ \beta & +i\ \text{sh}\ \beta \\ +i\ \text{sh}\ \beta & +i\ \text{sh}\ \beta & (\text{ch}\ \beta+1) & -(\text{ch}\ \beta-1) \\ -i\ \text{sh}\ \beta & -i\ \text{sh}\ \beta & -(\text{ch}\ \beta-1) & (\text{ch}\ \beta+1) \end{array} \right| ,$$

rotation $(0-3)$

$$U = \left| \begin{array}{rrrr} e^{-\beta} & 0 & 0 & 0 \\ 0 & e^{\beta} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right| ;$$

rotation $(2-3)$

$$U = \frac{1}{2} \left| \begin{array}{rrrr} (1+\cos\phi) & (1-\cos\phi) & i\sin\phi & -i\sin\phi \\ (1-\cos\phi) & (1+\cos\phi) & -i\sin\phi & i\sin\phi \\ i\sin\phi & -i\sin\phi & (1+\cos\phi) & (1-\cos\phi) \\ -i\sin\phi & i\sin\phi & (1-\cos\phi) & (1+\cos\phi) \end{array} \right| ,$$

rotation $(3-1)$

$$U = \frac{1}{2} \left| \begin{array}{rrrr} (1+\cos\phi) & (1-\cos\phi) & -\sin\phi & -\sin\phi \\ (1-\cos\phi) & (1+\cos\phi) & \sin\phi & \sin\phi \\ \sin\phi & -\sin\phi & (1+\cos\phi) & (1-\cos\phi) \\ \sin\frac{\phi}{2}\cos\frac{\phi}{2} & -\sin\frac{\phi}{2}\cos\frac{\phi}{2} & (1-\cos\phi) & (1+\cos\phi) \end{array} \right| ,$$

rotation $(1-2)$

$$U = \left| \begin{array}{rrrr} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{-i\phi} & 0 \\ 0 & 0 & 0 & e^{+i\phi} \end{array} \right| . \tag{231}$$

14 On finding parameters (k_a, k_a^*) from explicit form of the Lorentz matrices in isotropic form

To spinor transformation $B(k) \in SL(2.C)$

$$B(k) = \left| \begin{array}{cc} a & c \\ d & b \end{array} \right| , \qquad a\, b \; - \; c\, d = +1 \tag{232}$$

there corresponds a Lorentz matrix $L(k,\; k^*)$, function of $(a,\; b,\; c,\; d;\; a^*,\; b^*,\; c^*,\; d^*)$. After translating to isotropic representation $U(k,\; k^*) = S\; L(k, k^*)\; S^{-1}$, we get

$$U(k,\; k^*) = \left| \begin{array}{rrrr} b\, b^* & d\, d^* & -d\, b^* & -d\, b^* \\ c\, c^* & a\, a^* & -a\, c^* & -a^*\, c \\ -c\, b^* & -d^*\, a & a\, b^* & d^*\, c \\ -c^*\, b & -d\, a^* & d\, c^* & a^*\, b \end{array} \right| . \tag{233}$$

The problem reduces to finding complex $(a,\; b,\; c,\; d)$ from explicit form of $U(k,\; k^*)$. To this end, let us factorize each of these four parameters

$$a = \; A \; \exp^{i \; \arg(a)} = \; A\; \alpha\; , \qquad c = \; C \; \exp^{i \; \arg(c)} = \; C\; s\; ,$$
$$d = \; D \; \exp^{i \; arg(d)} = \; D\; t\; , \qquad b = \; B \; \exp^{i \; arg(b)} = \; B\; \beta\; .$$

so that

$$U(k,\; k^*) = \left| \begin{array}{cccc} B^2 & D^2 & -B\; D\; t/\beta & -B\; D\; \beta/t \\ C^2 & A^2 & -A\; C\; \alpha/s & -A\; C\; s/\alpha \\ -C\; B\; s/\beta & -A\; D\; \alpha/t & A\; B\; \alpha/\beta & C\; D\; s/t \\ -C\; B\; \beta/s & -A\; D\; t/\alpha & C\; D\; t/s & A\; B\; \beta/\alpha \end{array} \right| . \tag{234}$$

From relation det $\; B = +1$ it follows

$$AB = \alpha\beta \; \frac{s^2t^2 - 1}{s^2t^2 - \alpha^2\beta^2} \; , \qquad CD = st \; \frac{\alpha^2\beta^2 - 1}{s^2t^2 - \alpha^2\beta^2} \; . \tag{235}$$

Allowing for (235), for ${U_2}^2$ and ${U_2}^3$ we get

$${U_2}^2 = \alpha^2 \; [\; 1 + \frac{\alpha^2\beta^2 - 1}{s^2t^2 - \alpha^2\beta^2} \;] \; , \qquad {U_2}^3 = s^2 \; \frac{(\alpha^2\beta^2 - 1)}{s^2t^2 - \alpha^2\beta^2} \; . \tag{236}$$

and therefore

$${U_2}^2 = \alpha^2 + \frac{\alpha^2}{s^2} \; {U_2}^3 \quad , \qquad \frac{\alpha^2}{s^2} = \frac{{U_1}^2}{{U_1}^3}$$

so that

$$\alpha = \delta \; \sqrt{{U_2}^2 \; - \; \frac{{U_1}^2 \; {U_2}^3}{{U_1}^3}} \; , \qquad \delta = \pm 1 \; . \tag{237}$$

Having known α one determine remaining three phase factors: $s,\; \beta,\; t$:

$$s = -\alpha \; \frac{A\; C}{{U_1}^2} \; , \quad \beta = s \; \frac{{U_3}^1}{C\; B} \; , \qquad t = -\beta \; \frac{B\; D}{{U_0}^3} \; . \tag{238}$$

besides $A,\ B,\ C,\ D$ are given by

$$A = \sqrt{U_1{}^1}\ , \quad B = \sqrt{U_0{}^0}\ ,$$
$$C = \sqrt{U_1{}^0}\ , \quad D = \sqrt{U_0{}^1}\ . \qquad (239)$$

Two special case should be considered separately:

<u>$c = d = 0$</u>

$$A = \sqrt{U_1{}^1}\ , \quad B = \sqrt{U_0{}^0}\ , \quad A\ B = 1\ ,$$
$$\alpha = \delta\ \sqrt{U_2{}^2}\ , \qquad \beta = +\frac{1}{\alpha}\ ; \qquad (240)$$

<u>$a = b = 0$</u>

$$C = \sqrt{U_1{}^0}\ , \quad D = \sqrt{U_0{}^1}\ , \quad C\ D = 1\ ,$$
$$s = \delta\ \sqrt{U_2{}^3}\ , \quad t = -\frac{1}{s}\ . \qquad (241)$$

15 Spinor representation for electromagnetic 4-potential and tensor, Stokes 4-vector and 2-rank tensor for a completely polarized light

Let start with the well-known relations between 2-rank bi-spinors and simplest tensors. Bi-spinor of second rang $U = \Psi \otimes \Psi$ can be resolved into scalar Φ, vector Φ_b ; pseudoscalar $\tilde{\Phi}$, pseudovector $\tilde{\Phi}_b$, and antisymmetric tensor Φ_{ab} [4]

$$U = \Psi \otimes \Psi = \Big[-i\ \Phi + \gamma^b\ \Phi_b + i\ \sigma^{ab}\ \Phi_{ab} + \gamma^5\ \tilde{\Phi} + i\ \gamma^b \gamma^5\ \tilde{\Phi}_b \Big]\ E^{-1}\ ; \qquad (242)$$

let us refer all consideration to the spinor basis

$$U = \left| \begin{array}{cc} \xi^\alpha \xi^\beta & \xi^\alpha \eta_{\dot\beta} \\ \eta_{\dot\alpha} \xi^\beta & \eta_{\dot\alpha} \eta_{\dot\beta} \end{array} \right| = \left| \begin{array}{cc} \xi^{\alpha\beta} & \Delta^{\alpha}{}_{\dot\beta} \\ H_{\dot\alpha}{}^{\beta} & \eta_{\dot\alpha\dot\beta} \end{array} \right| , \qquad \gamma^a = \left| \begin{array}{cc} 0 & \bar\sigma^a \\ \sigma^a & 0 \end{array} \right| ,$$
$$\sigma^{ab} = \frac{1}{4} \left| \begin{array}{cc} (\bar\sigma^a \sigma^b - \bar\sigma^b \sigma^a) & 0 \\ 0 & (\sigma^a \bar\sigma^b - \sigma^b \bar\sigma^a) \end{array} \right| = \left| \begin{array}{cc} \Sigma^{ab} & 0 \\ 0 & \bar\Sigma^{ab} \end{array} \right| ,$$
$$\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3 = \left| \begin{array}{cc} -I & 0 \\ 0 & +I \end{array} \right| ; \qquad (243)$$

E stands for a bi-spinor metric matrix

$$E = \left| \begin{array}{cc} \epsilon & 0 \\ 0 & \dot\epsilon^{-1} \end{array} \right| = \left| \begin{array}{cc} \epsilon_{\alpha\beta} & 0 \\ 0 & \epsilon^{\dot\alpha\dot\beta} \end{array} \right| = \left| \begin{array}{cc} i\sigma^2 & 0 \\ 0 & -i\sigma^2 \end{array} \right| ,$$
$$E^2 = -I\ , \qquad \tilde{E} = -E\ , \qquad \mathrm{Sp}\ E = 0\ , \qquad \tilde{\sigma}^{ab}\ E = -E\ \sigma^{ab}\ . \qquad (244)$$

[4] In this section we use Dirac matrices instead of $\alpha_a, \beta_a, \alpha_a\beta_a$; for more details on connection between them see in [10].

Inverse to (242) relations look

$$\Phi_a = \frac{1}{4}\,\mathrm{Sp}\,[E\gamma_a U]\,, \qquad \tilde{\Phi}_a = \frac{1}{4i}\,\mathrm{Sp}\,[E\gamma^5\gamma_a U]\,,$$
$$\Phi = \frac{i}{4}\,\mathrm{Sp}\,[EU]\,, \qquad \tilde{\Phi} = \frac{1}{4}\,\mathrm{Sp}\,[E\gamma^5 U]\,,$$
$$\Phi_{mn} = -\frac{1}{2i}\,\mathrm{Sp}\,[E\sigma_{mn}U]\,. \tag{245}$$

More insight can be reached in 2-spinor basis:

$$\left|\begin{array}{cc} \xi^{\alpha\beta} & \Delta^{\alpha}{}_{\dot\beta} \\ H_{\dot\alpha}{}^{\beta} & \eta_{\dot\alpha\dot\beta} \end{array}\right| = \left|\begin{array}{cc} [\,(-i\,\Phi - \tilde{\Phi}) + i\,\Sigma^{mn}\,\Phi_{mn}\,]\,\epsilon^{-1} & \bar{\sigma}^{l}\,(\Phi_l + i\,\tilde{\Phi}_l)\,\dot{\epsilon} \\ \sigma^{l}\,(\Phi_l - i\,\tilde{\Phi}_l)\,\epsilon^{-1} & [\,(-i\,\Phi + \tilde{\Phi}) + i\,\bar{\Sigma}^{mn}\,\Phi_{mn}\,]\,\dot{\epsilon} \end{array}\right|\,. \tag{246}$$

For vector and pseudovector we have

$$\frac{1}{2}\,\mathrm{Sp}\,(\dot{\epsilon}^{-1}\,\sigma_l\Delta) = \frac{1}{2}\;\mathrm{Sp}\,[\,\sigma_l\bar{\sigma}^k\,(\Phi_k + i\,\tilde{\Phi}_k)\,] = \Phi_l + i\,\tilde{\Phi}_l\,,$$
$$\frac{1}{2}\,\mathrm{Sp}\,(\epsilon\,\bar{\sigma}_l H) = \frac{1}{2}\;\mathrm{Sp}\,[\,\bar{\sigma}_l\,\sigma^k\,(\Phi_k - i\,\tilde{\Phi}_k)\,] = \Phi_l - i\,\tilde{\Phi}_l\,,$$

so that

$$\frac{1}{4}\,\mathrm{Sp}\,(E\gamma_l U) = \frac{1}{4}\;\Big[\,\mathrm{Sp}\,(\dot{\epsilon}^{-1}\,\sigma_l\Delta) + \mathrm{Sp}\,(\epsilon\,\bar{\sigma}_l H)\,\Big] = \Phi_l\,,$$
$$\frac{1}{4i}\,\mathrm{Sp}\,(E\gamma^5\gamma_l U) = \frac{1}{4i}\;\Big[\,\mathrm{Sp}\,(\dot{\epsilon}^{-1}\,\sigma_l\Delta) - \mathrm{Sp}\,(\epsilon\,\bar{\sigma}_l H)\,\Big] = \tilde{\Phi}_l\,. \tag{247}$$

For scalar and pseudoscalar we have

$$\frac{1}{2}\,\mathrm{Sp}\,(\epsilon\,\xi) = -i\,\Phi - \tilde{\Phi}\,, \qquad \frac{1}{2}\,\mathrm{Sp}\,(\dot{\epsilon}^{-1})\,\eta = -i\,\Phi + \tilde{\Phi}$$

so that

$$\frac{i}{4}\,\mathrm{Sp}\,(EU) = \frac{1}{4}\;\Big[\,\mathrm{Sp}\,(\epsilon\,\xi) + \mathrm{Sp}\,(\dot{\epsilon}^{-1}\,\eta)\,\Big] = \Phi\,,$$
$$\frac{1}{4}\,\mathrm{Sp}\,(E\gamma^5 U) = \frac{1}{4}\;\Big[\,\mathrm{Sp}\,(\dot{\epsilon}^{-1}\,\eta) - \mathrm{Sp}\,(\epsilon\,\xi)\,\Big] = \tilde{\Phi}\,. \tag{248}$$

At last, with the help of formulas

$$\mathrm{Sp}\,(\Sigma^{kl}\Sigma^{mn}) = \frac{1}{2}\;\Big[(g^{kn}g^{lm} - g^{ln}g^{km}) - i\,\epsilon^{klmn}\Big]\,,$$
$$\mathrm{Sp}\,(\bar{\Sigma}^{kl}\bar{\Sigma}^{mn}) = \frac{1}{2}\;\Big[(g^{kn}g^{lm} - g^{ln}g^{km}) + i\,\epsilon^{klmn}\Big]$$

we get

$$\mathrm{Sp}\,(\epsilon\,\Sigma^{mn}\,\xi) = -i\,\Phi^{mn} + \frac{1}{2}\,\epsilon^{mnkl}\,\Phi_{kl}\,,$$
$$\mathrm{Sp}\,(\dot{\epsilon}^{-1}\,\bar{\Sigma}^{mn}\,\eta) = -i\,\Phi^{mn} - \frac{1}{2}\,\epsilon^{mnkl}\,\Phi_{kl}$$

so that

$$\frac{i}{2}\ \mathrm{Sp}\ (E\sigma^{mn}U) = \frac{i}{2}\ \Big[\ \mathrm{Sp}\ (\dot{\epsilon}^{-1}\ \bar{\Sigma}^{mn}\ \eta) + \mathrm{Sp}\ (\epsilon\ \Sigma^{mn}\ \xi)\ \Big] = \Phi^{mn}\ ,$$

$$\frac{1}{2i}\ \mathrm{Sp}\ (E\gamma^{5}\sigma^{mn}U) = \frac{1}{2i}\ \Big[\ \mathrm{Sp}\ (\dot{\epsilon}^{-1}\ \bar{\Sigma}^{mn}\ \eta) - \mathrm{Sp}\ (\epsilon\ \Sigma^{mn}\ \xi)\ \Big] =$$

$$= +\frac{i}{2}\ \epsilon^{mnkl}\ \Phi_{kl} = \tilde{\Phi}^{mn}\ . \qquad (249)$$

First, we are interested in two vectors obtained from spinors:

$$\Phi_a = \frac{1}{2}\mathrm{Sp}\left| \begin{array}{cc} i\sigma^2\bar{\sigma}_a H & i\sigma^2\bar{\sigma}_a\eta \\ -i\sigma^2\sigma_a\xi & -i\sigma^2\sigma_a\Delta \end{array} \right|$$

so that

$$\Phi_0 = \frac{1}{2}\mathrm{Sp}\left| \begin{array}{cc} i\sigma^2 H & ... \\ ... & -i\sigma^2\Delta \end{array} \right| = \frac{1}{2}\ [\ (H_{\dot{2}}{}^{1} - H_{\dot{1}}{}^{2}) - (\Delta^{2}{}_{\dot{1}} - \Delta^{1}{}_{\dot{2}})\ ] = \xi^1\eta_{\dot{2}} - \xi^2\eta_{\dot{1}}\ ,$$

$$\Phi_1 = \frac{1}{2}\mathrm{Sp}\left| \begin{array}{cc} i\sigma^2\sigma^1 H & ... \\ ... & i\sigma^2\sigma^1\Delta \end{array} \right| = \frac{1}{2}\ [\ (H_{\dot{1}}{}^{1} - H_{\dot{2}}{}^{2}) + (\Delta^{1}{}_{\dot{1}} - \Delta^{2}{}_{\dot{2}})\ ] = \xi^1\eta_{\dot{1}} - \xi^2\eta_{\dot{2}}\ ,$$

$$\Phi_2 = \frac{1}{2}\mathrm{Sp}\left| \begin{array}{cc} i\sigma^2\sigma^2 H & ... \\ ... & i\sigma^2\sigma^2\Delta \end{array} \right| = \frac{i}{2}\ [\ (H_{\dot{1}}{}^{1} + H_{\dot{2}}{}^{2}) + (\Delta^{1}{}_{\dot{1}} + \Delta^{2}{}_{\dot{2}})\ ] = i\ (\xi^1\eta_{\dot{1}} + \xi^2\eta_{\dot{2}})\ ,$$

$$\Phi_3 = \frac{1}{2}\mathrm{Sp}\left| \begin{array}{cc} i\sigma^2\sigma^3 H & ... \\ ... & i\sigma^2\sigma^3\Delta \end{array} \right| = \frac{-1}{2}\ [\ (H_{\dot{2}}{}^{1} + H_{\dot{1}}{}^{2}) + (\Delta^{2}{}_{\dot{1}} + \Delta^{1}{}_{\dot{2}})\ ] = -\ (\xi^1\eta_{\dot{2}} + \xi^2\eta_{\dot{1}})\ ;$$

and for pseudovector

$$\tilde{\Phi}_a = \frac{1}{2}\mathrm{Sp}\left| \begin{array}{cc} -i\sigma^2\bar{\sigma}_a H & -i\sigma^2\bar{\sigma}_a\eta \\ -i\sigma^2\sigma_a\xi & -i\sigma^2\sigma_a\Delta \end{array} \right|\ ,$$

so that

$$\tilde{\Phi}_0 = \frac{1}{2}\ \mathrm{Sp}\ \left| \begin{array}{cc} -i\ \sigma^2\ H & ... \\ ... & -i\ \sigma^2\ \Delta \end{array} \right| = \frac{1}{2}\ [\ -(H_{\dot{2}}{}^{1} - H_{\dot{1}}{}^{2}) - (\Delta^{2}{}_{\dot{1}} - \Delta^{1}{}_{\dot{2}})\ ] = 0\ ,$$

$$\tilde{\Phi}_1 = \frac{1}{2}\mathrm{Sp}\left| \begin{array}{cc} -i\sigma^2\sigma^1 H & ... \\ ... & i\sigma^2\sigma^1\Delta \end{array} \right| = \frac{1}{2}\ [\ -(H_{\dot{1}}{}^{1} - H_{\dot{2}}{}^{2}) + (\Delta^{1}{}_{\dot{1}} - \Delta^{2}{}_{\dot{2}})\ ] = 0\ ,$$

$$\tilde{\Phi}_2 = \frac{1}{2}\mathrm{Sp}\left| \begin{array}{cc} -i\sigma^2\sigma^2 H & ... \\ ... & i\sigma^2\sigma^2\Delta \end{array} \right| = \frac{i}{2}\ [\ -(H_{\dot{1}}{}^{1} + H_{\dot{2}}{}^{2}) + (\Delta^{1}{}_{\dot{1}} + \Delta^{2}{}_{\dot{2}})\ ] = 0\ ,$$

$$\tilde{\Phi}_3 = \frac{1}{2}\mathrm{Sp}\left| \begin{array}{cc} -i\sigma^2\sigma^3 H & ... \\ ... & i\sigma^2\sigma^3\Delta \end{array} \right| = \frac{-1}{2}\ [\ -(H_{\dot{2}}{}^{1} + H_{\dot{1}}{}^{2}) + (\Delta^{2}{}_{\dot{1}} + \Delta^{1}{}_{\dot{2}})\ ] = 0\ .$$

In the same manner we get for scalar and pseudoscalar:

$$\Phi = \frac{i}{4}\ \mathrm{Sp}\ EU = \frac{i}{4}\ \mathrm{Sp}\ \left| \begin{array}{cc} +i\sigma^2\xi & ... \\ ... & -i\sigma^2\eta \end{array} \right| = \frac{i}{4}\ [\ +(\xi^{21} - \xi^{12}) - (\eta_{\dot{2}\dot{1}} + \eta_{\dot{1}\dot{2}})\ ] = 0\ ,$$

$$\tilde{\Phi} = \frac{i}{4}\ \mathrm{Sp}\ E\gamma^5 U = \frac{i}{4}\ \mathrm{Sp}\ \left| \begin{array}{cc} -i\sigma^2\xi & ... \\ ... & -i\sigma^2\eta \end{array} \right| = \frac{i}{4}\ [\ -(\xi^{21} - \xi^{12}) - (\eta_{\dot{2}\dot{1}} + \eta_{\dot{1}\dot{2}})\ ] = 0\ ;$$

and for antisymmetric tensor

$$\Phi^{mn} = -\frac{1}{2i}\,\text{Sp}\,[\,E\sigma^{mn}U\,] = -\frac{1}{2i}\,\text{Sp}\left|\begin{array}{cc} i\sigma^2\Sigma^{mn}\xi & ... \\ ... & -i\sigma^2\bar{\Sigma}^{mn}\eta \end{array}\right| ,$$

so that

$$\Phi^{01} = \frac{i}{4}\,\text{Sp}\left|\begin{array}{cc} \sigma^3\xi & ... \\ ... & \sigma^3\eta \end{array}\right| =$$

$$= \frac{i}{4}\,[\,(\xi^{11} - \xi^{22}) + (\eta_{\dot{1}\dot{1}} - \eta_{\dot{2}\dot{2}})\,] = \frac{i}{4}\,[\,(\xi^1\xi^1 - \xi^2\xi^2) + (\eta_{\dot{1}}\eta_{\dot{1}} - \eta_{\dot{2}}\eta_{\dot{2}})\,] ,$$

$$\Phi^{23} = \frac{1}{4}\,\text{Sp}\left|\begin{array}{cc} \sigma^3\xi & ... \\ ... & -\sigma^3\eta \end{array}\right| =$$

$$= \frac{1}{4}\,[\,(\xi^{11} - \xi^{22}) - (\eta_{\dot{1}\dot{1}} - \eta_{\dot{2}\dot{2}})\,] = \frac{1}{4}\,[\,(\xi^1\xi^1 - \xi^2\xi^2) - (\eta_{\dot{1}}\eta_{\dot{1}} - \eta_{\dot{2}}\eta_{\dot{2}})\,] ,$$

$$\Phi^{02} = -\frac{1}{4}\,\text{Sp}\left|\begin{array}{cc} \xi & ... \\ ... & \eta \end{array}\right| =$$

$$= -\frac{1}{4}\,[\,(\xi^{11} + \xi^{22}) + (\eta_{\dot{1}\dot{1}} + \eta_{\dot{2}\dot{2}})\,] = -\frac{1}{4}\,[\,(\xi^1\xi^1 + \xi^2\xi^2) + (\eta_{\dot{1}}\eta_{\dot{1}} + \eta_{\dot{2}}\eta_{\dot{2}})\,] ,$$

$$\Phi^{31} = -\frac{1}{4i}\,\text{Sp}\left|\begin{array}{cc} \xi & ... \\ ... & -\eta \end{array}\right| =$$

$$= -\frac{1}{4i}\,[\,(\xi^{11} + \xi^{22}) - (\eta_{\dot{1}\dot{1}} + \eta_{\dot{2}\dot{2}})\,] = -\frac{1}{4i}\,[\,(\xi^1\xi^1 + \xi^2\xi^2) - (\eta_{\dot{1}}\eta_{\dot{1}} + \eta_{\dot{2}}\eta_{\dot{2}})\,] ,$$

$$\Phi^{03} = -\frac{i}{4}\,\text{Sp}\left|\begin{array}{cc} \sigma^1\xi & ... \\ ... & \sigma^1\eta \end{array}\right| = -\frac{i}{4}\,[\,(\xi^{21} + \xi^{12}) + (\eta_{\dot{2}\dot{1}} + \eta_{\dot{1}\dot{2}})] = -\frac{i}{2}\,[\,\xi^1\xi^2 + \eta_{\dot{1}}\eta_{\dot{2}}] ,$$

$$\Phi^{12} = -\frac{1}{4}\,\text{Sp}\left|\begin{array}{cc} \sigma^1\xi & ... \\ ... & -\sigma^1\eta \end{array}\right| = -\frac{1}{4}\,[\,(\xi^{21} + \xi^{12}) - (\eta_{\dot{2}\dot{1}} + \eta_{\dot{1}\dot{2}})] = -\frac{1}{2}\,[\,\xi^1\xi^2 - \eta_{\dot{1}}\eta_{\dot{2}}] ,$$

Collecting results together

$$\Psi = \left|\begin{array}{c} \xi^\alpha \\ \eta_{\dot{\alpha}} \end{array}\right| , \qquad \Psi\otimes\Psi \qquad \Longrightarrow \qquad \Phi = 0, \;\; \tilde{\Phi} = 0, \;\; \tilde{\Phi}_a = 0, \;\; \Phi_a \neq 0, \;\; \Phi_{mn} \neq 0 ,$$

we see that to have real vector and tensor one must impose additional restriction

$$\eta = +i\,\sigma^2\,\xi^* \qquad \Longrightarrow \qquad \eta_{\dot{1}} = +\,\xi^{2*} , \;\; \eta_{\dot{2}} = -\,\xi^{1*} ; \tag{250}$$

which result in

$$\Phi_0 = -(\xi^1\,\xi^{1*} + \xi^2\,\xi^{2*}) < 0 , \qquad \Phi_3 = (\xi^1\,\xi^{1*} - \xi^2\,\xi^{2*}) ,$$
$$\Phi_1 = (\xi^1\,\xi^{2*} + \xi^2\,\xi^{1*}) , \qquad \Phi_2 = i\,(\xi^1\,\xi^{2*} - \xi^2\,\xi^{1*}) ;$$

$$\Phi^{01} = \frac{i}{4}\,[\,(\xi^1\,\xi^1 - \xi^2\,\xi^2) + (\xi^{2*}\,\xi^{2*} - \xi^{1*}\,\xi^{1*})\,]\,,$$
$$\Phi^{23} = \frac{1}{4}\,[\,(\xi^1\,\xi^1 - \xi^2\,\xi^2) - (\xi^{2*}\,\xi^{2*} - \xi^{1*}\,\xi^{1*})\,]\,,$$

$$\Phi^{02} = -\frac{1}{4}\,[\,(\xi^1\,\xi^1 + \xi^2\,\xi^2) + (\xi^{2*}\,\xi^{2*} + \xi^{1*}\,\xi^{1*})\,]\,,$$
$$\Phi^{31} = -\frac{1}{4i}\,[\,(\xi^1\,\xi^1 + \xi^2\,\xi^2) - (\xi^{2*}\,\xi^{2*} + \xi^{1*}\,\xi^{1*})\,]\,,$$

$$\Phi^{03} = -\frac{i}{2}\,(\,\xi^1\,\xi^2 - \xi^{2*}\,\xi^{1*})\,, \qquad \Phi^{12} = -\frac{1}{2}\,[\,\xi^1\,\xi^2 + \xi^{2*}\,\xi^{1*}]\,, \tag{251}$$

There exists alternative additional restriction:

$$\eta = -i\,\sigma^2\,\xi^* \qquad \Longrightarrow \qquad \eta_{\dot 1} = -\xi^{2*}\,, \;\; \eta_{\dot 2} = +\xi^{1*}\,; \tag{252}$$

which result in (compare with (251))

$$\Phi_0 = (\xi^1\,\xi^{1*} + \xi^2\,\xi^{2*}) > 0\,, \qquad \Phi_3 = -(\xi^1\,\xi^{1*} - \xi^2\,\xi^{2*})\,,$$
$$\Phi_1 = -(\xi^1\,\xi^{2*} + \xi^2\,\xi^{1*})\,, \qquad \Phi_2 = -i\,(\xi^1\,\xi^{2*} - \xi^2\,\xi^{1*})\,;$$

$$\Phi^{01} = \frac{i}{4}\,[\,(\xi^1\,\xi^1 - \xi^2\,\xi^2) + (\xi^{2*}\,\xi^{2*} - \xi^{1*}\,\xi^{1*})\,]\,,$$
$$\Phi^{23} = \frac{1}{4}\,[\,(\xi^1\,\xi^1 - \xi^2\,\xi^2) - (\xi^{2*}\,\xi^{2*} - \xi^{1*}\,\xi^{1*})\,]\,,$$

$$\Phi^{02} = -\frac{1}{4}\,[\,(\xi^1\,\xi^1 + \xi^2\,\xi^2) + (\xi^{2*}\,\xi^{2*} + \xi^{1*}\,\xi^{1*})\,]\,,$$
$$\Phi^{31} = -\frac{1}{4i}\,[\,(\xi^1\,\xi^1 + \xi^2\,\xi^2) - (\xi^{2*}\,\xi^{2*} + \xi^{1*}\,\xi^{1*})\,]\,,$$

$$\Phi^{03} = -\frac{i}{2}\,(\,\xi^1\,\xi^2 - \xi^{2*}\,\xi^{1*})\,, \qquad \Phi^{12} = -\frac{1}{2}\,[\,\xi^1\,\xi^2 + \xi^{2*}\,\xi^{1*}]\,, \tag{253}$$

The last case (252) – (252) seems to be appropriate to describe Stokes 4-vector and determine Stokes 2-rank tensor:

$$\Psi = \left|\begin{array}{c} \xi \\ \eta = -i\,\sigma^2\,\xi^* \end{array}\right|\,, \qquad \Psi \otimes \Psi \qquad \Longrightarrow \qquad S_a \neq 0, \;\; S_{mn} \neq 0\,,$$

$$S_0 = (\xi^1\,\xi^{1*} + \xi^2\,\xi^{2*}) > 0\,, \qquad S_3 = -(\xi^1\,\xi^{1*} - \xi^2\,\xi^{2*})\,,$$
$$S_1 = -(\xi^1\,\xi^{2*} + \xi^2\,\xi^{1*})\,, \qquad S_2 = -i\,(\xi^1\,\xi^{2*} - \xi^2\,\xi^{1*})\,;$$

$$a^1 = S^{01} = \frac{i}{4} \, [\, (\xi^1 \, \xi^1 - \xi^2 \, \xi^2) + (\xi^{2*} \, \xi^{2*} - \xi^{1*} \, \xi^{1*}) \,] \, ,$$
$$b^1 = S^{23} = \frac{1}{4} \, [\, (\xi^1 \, \xi^1 - \xi^2 \, \xi^2) - (\xi^{2*} \, \xi^{2*} - \xi^{1*} \, \xi^{1*}) \,] \, ,$$

$$a^2 = S^{02} = -\frac{1}{4} \, [\, (\xi^1 \, \xi^1 + \xi^2 \, \xi^2) + (\xi^{2*} \, \xi^{2*} + \xi^{1*} \, \xi^{1*}) \,] \, ,$$
$$b^2 = S^{31} = -\frac{1}{4i} \, [\, (\xi^1 \, \xi^1 + \xi^2 \, \xi^2) - (\xi^{2*} \, \xi^{2*} + \xi^{1*} \, \xi^{1*}) \,] \, ,$$

$$a^3 = S^{03} = -\frac{i}{2} \, (\, \xi^1 \, \xi^2 - \xi^{2*} \, \xi^{1*}) \, , \qquad b^3 = S^{12} = -\frac{1}{2} \, (\, \xi^1 \, \xi^2 + \xi^{2*} \, \xi^{1*}) \, . \tag{254}$$

Let us calculate the main invariant

$$\begin{gathered} S_0 S_0 - S_j S_j = \\ = (\xi^1 \, \xi^{1*} + \xi^2 \, \xi^{2*})^2 - (\xi^1 \, \xi^{1*} - \xi^2 \, \xi^{2*})^2 - \\ -(\xi^1 \, \xi^{2*} + \xi^2 \, \xi^{1*})^2 + (\xi^1 \, \xi^{2*} - \xi^2 \, \xi^{1*})^2 = \\ = 4(\xi^1 \xi^{1*}) \, (\xi^2 \xi^{2*}) - 4(\xi^1 \xi^{2*}) \, (\xi^2 \xi^{1*}) = 0 \, ; \end{gathered}$$

so S_a may be considered as a Stokes 4-vector for a completely polarized light. 4-tensor S_{mn}, being constructed from Jones bi-spinor Ψ, is a Stokes 2-rank tensor.

Let us calculate two invariants for S_{mn}. The first is

$$\begin{gathered} I_1 = -\frac{1}{2} \, S^{mn} S_{mn} = \mathbf{a}^2 - \mathbf{b}^2 = \frac{1}{16} \times \{ \\ -[\, (\xi^1 \, \xi^1 - \xi^2 \, \xi^2) + (\xi^{2*} \, \xi^{2*} - \xi^{1*} \, \xi^{1*}) \,]^2 - [\, (\xi^1 \, \xi^1 - \xi^2 \, \xi^2) - (\xi^{2*} \, \xi^{2*} - \xi^{1*} \, \xi^{1*}) \,]^2 + \\ +[\, (\xi^1 \, \xi^1 + \xi^2 \, \xi^2) + (\xi^{2*} \, \xi^{2*} + \xi^{1*} \, \xi^{1*}) \,]^2 + [\, (\xi^1 \, \xi^1 + \xi^2 \, \xi^2) - (\xi^{2*} \, \xi^{2*} + \xi^{1*} \, \xi^{1*}) \,]^2 + \\ -4(\, \xi^1 \, \xi^2 - \xi^{2*} \, \xi^{1*})^2 - 4(\, \xi^1 \, \xi^2 + \xi^{2*} \, \xi^{1*})^2 = \\ = \frac{1}{16} \, [\, -2(\xi^1 \, \xi^1 - \xi^2 \, \xi^2)^2 - 2(\xi^{1*} \, \xi^{1*} - \xi^{2*} \, \xi^2 *) + \\ +2(\xi^1 \, \xi^1 + \xi^2 \, \xi^2)^2 + 2(\xi^{1*} \, \xi^{1*} + \xi^{2*} \, \xi^2 *) - \\ -8(\xi^1)^2 (\xi^2)^2 - 8(\xi^{1*})^2 (\xi^{2*})^2 \,] \, , \end{gathered}$$

that is

$$I_1 = -\frac{1}{2} \, S^{mn} S_{mn} = 0 \, . \tag{255}$$

The second invariant is

$$\begin{gathered} I_2 = \frac{1}{4} \, \epsilon_{abmn} S^{ab} S^{mn} = \mathbf{ab} = \\ \frac{i}{16} \, [\, (\xi^1 \, \xi^1 - \xi^2 \, \xi^2)^2 - (\xi^{1*} \, \xi^{1*} - \xi^{2*} \, \xi^{2*})^2 - \\ -(\xi^1 \, \xi^1 + \xi^2 \, \xi^2)^2 + (\xi^{1*} \, \xi^{1*} + \xi^{2*} \, \xi^{2*})^2 + \\ +4(\, \xi^1 \, \xi^2 - \xi^{2*} \, \xi^{1*})(\, \xi^1 \, \xi^2 + \xi^{2*} \, \xi^{1*}) \,] = \\ = \frac{i}{16} \, [\, (-4\xi^1 \xi^1 \xi^2 \xi^2 + 4\xi^{1*} \xi^{1*} \xi^{2*} \xi^{2*}) + (4\xi^1 \xi^1 \xi^2 \xi^2 - 4\xi^{1*} \xi^{1*} \xi^{2*} \xi^{2*}) \,] = 0 \, . \end{gathered}$$

so that

$$I_2 = \frac{1}{4}\,\epsilon_{abmn} S^{ab} S^{mn} = \mathbf{ab} = 0\;. \tag{256}$$

Finally, let specify Stokes 4-vector and 4-tensor to $(M, N, \Delta = \alpha - \beta)$ parameters:

$$\Psi = \left| \begin{array}{r} N\;e^{i\alpha} \\ +M\;e^{i\beta} \\ -M\;e^{-i\beta} \\ N\;e^{-i\alpha} \end{array} \right| , \qquad \Psi \otimes \Psi \qquad \Longrightarrow \qquad S_a \neq 0, \;\; S_{mn} \neq 0\;,$$

$$S_0 = M^2 + N^2\;, \qquad S_3 = M^2 - N^2\;,$$
$$S_1 = -2MN\cos(\alpha - \beta)\;, \qquad S_2 = 2MN\sin(\alpha - \beta)$$

which coincides with (65); and

$$\begin{aligned}
a^1 = S^{01} &= -\frac{1}{2}(N^2\sin 2\alpha - M^2\sin 2\beta)\;,\\
b^1 = S^{23} &= +\frac{1}{2}(N^2\cos 2\alpha - M^2\cos 2\beta)\;,\\
a^2 = S^{02} &= -\frac{1}{2}(N^2\cos 2\alpha + M^2\cos 2\beta)\;,\\
b^2 = S^{31} &= -\frac{1}{2}(N^2\sin 2\alpha + M^2\sin 2\beta)\;,\\
a^3 = S^{03} &= +NM\sin(\alpha + \beta)\;,\\
b^3 = S^{12} &= -NM\cos(\alpha + \beta)\;.
\end{aligned} \tag{257}$$

Two vectors $\mathbf{a}, \mathbf{b}$ are determined by 4 parameters N, M, α, β, additional identities hold

$$\mathbf{a}^2 = \mathbf{b}^2 = \frac{(N^2 + M^2)^2}{4}\;, \qquad \mathbf{ab} = 0\;;$$

therefore the quantities $\mathbf{a}, \mathbf{b}$ depend in fact upon 4 independent parameters $N, M, \beta - \alpha, \beta + \alpha$; whereas Stokes 4-vector depends upon only three ones $N, M, \beta - \alpha$.

Instead of Stokes 4-tensor S_{ab} one may introduce a complex Stokes 3-vector $\mathbf{s} = \mathbf{a} + i\mathbf{b}$ with the following explicit components (see (254)):

$$\begin{aligned}
s^1 = a^1 + ib^1 = S^{01} + iS^{23} &= \frac{i}{2}\;(\xi^1\;\xi^1 - \xi^2\;\xi^2)\;,\\
s^2 = a^2 + ib^2 = S^{02} + iS^{31} &= -\frac{1}{2}\;(\xi^1\;\xi^1 + \xi^2\;\xi^2)\\
s^3 = a^3 + ib^3 = S^{03} = iS^{12} &= -i\;\xi^1\;\xi^2\;.
\end{aligned} \tag{258}$$

from whence it follows

$$s_1 + is_2 = -i\;\xi^2\xi^2\;, \qquad s_1 - is_2 = +i\;\xi^1\xi^1\;, \qquad s^3 - i\;\;\xi^1\;\xi^2\;. \tag{259}$$

The quantity $\mathbf{s}$ transforms as a vector under complex rotation group $SO(3.C)$. The later permits to introduce additionally to Jones spinor and Mueller vector formalisms one other technique based on the use of complex 3-vector under complex rotation group $SO(3.C)$:

$$\mathbf{s} = \mathbf{a} + i\mathbf{b} = \frac{1}{2} \left| \begin{array}{c} i\,(N^2 e^{2i\alpha} - M^2 e^{2i\beta}) \\ -\,(N^2 e^{2i\alpha} + M^2 e^{2i\beta}) \\ -2i\; NM\; e^{i(\alpha+\beta)} \end{array} \right| , \tag{260}$$

evidently this complex vector is isotropic $\mathbf{s}^2 = 0$, the later condition provide us with two additional condition, so $\mathbf{s}$ depends on 4 parameters.

16 Spinor representation for a space-time vectors

The question relevant to description of a partly polarized in terms of Jones spinor object still remains unsolved. Let us turn back and consider possibility to construct vector and tensor in terms of spinor components with no additional restriction on bi-spinor (like $\eta = \pm i\sigma^2\xi^*$):

$$\Phi_0 = \xi^1\eta_{\dot{2}} - \xi^2\eta_{\dot{1}} \; , \qquad \Phi_3 = -\,(\xi^1\eta_{\dot{2}} + \xi^2\eta_{\dot{1}}) \; ,$$
$$\Phi_1 = \xi^1\eta_{\dot{1}} - \xi^2\eta_{\dot{2}} \; , \qquad \Phi_2 = i\,(\xi^1\eta_{\dot{1}} + \xi^2\eta_{\dot{2}}) \; ;$$

$$\Phi_0^2 - \Phi_3^2 = (\xi^1\eta_{\dot{2}} - \xi^2\eta_{\dot{1}})^2 - (\xi^1\eta_{\dot{2}} + \xi^2\eta_{\dot{1}})^2 = -4\,\xi^1\xi^2\;\eta_{\dot{1}}\eta_{\dot{2}} \; ,$$
$$\Phi_1^2 + \Phi_2^2 = (\xi^1\eta_{\dot{1}} - \xi^2\eta_{\dot{2}})^2 - (\xi^1\eta_{\dot{1}} + \xi^2\eta_{\dot{2}})^2 = -4\,\xi^1\xi^2\;\eta_{\dot{1}}\eta_{\dot{2}} \; ,$$

$$\Phi^0\Phi^0 - \Phi_1\Phi^j - \Phi_2\Phi^2 - \Phi_3\Phi^3 = 0 \; ; \tag{261}$$

the complex vector Φ_a is isotropic. Let us separate real and imaginary parts:

$$\Phi_0 = A + iB \; , \qquad \Phi_j = A_j + iB_j \; ,$$
$$A^2 - \mathbf{A}^2 = B^2 - \mathbf{B}^2, \qquad AB - \mathbf{A}\mathbf{B} = 0 \; . \tag{262}$$

So two real 4-vectors A_n and B_n have the same length, they are orthogonal to each other, and they may be non-isotropic ones.

The main relationships between spinor and tensors are

$$E\;(\Psi \otimes \Psi) = \gamma^n\;\Phi_n + i\;\sigma^{mn}\;\Phi_{mn}$$
$$E\;(\Psi^* \otimes \Psi^*) = (\gamma^n)^*\;\Phi_n^* - i\;(\sigma^{mn})^*\;\Phi_{mn}^* \; . \tag{263}$$

As in spinor basis we have identities

$$(\gamma^n)^* = \gamma^2\;\gamma^n\;\gamma^2 \; , \qquad (\sigma^{mn})^* = -\gamma^2\;\sigma^{mn}\;\gamma^2 \; ,$$

relations (263) read

$$E\;(\Psi \otimes \Psi) = \gamma^n\;\Phi_n + i\;\sigma^{mn}\;\Phi_{mn}$$
$$\gamma^2\;E\;(\Psi^* \otimes \Psi^*)\gamma^2 = \gamma^n\;\Phi_n^* + i\;\sigma^{mn}\;\Phi_{mn}^* \; . \tag{264}$$

and further

$$
\begin{aligned}
(\Psi \otimes \Psi) &= [\, \gamma^n \, \Phi_n + i \, \sigma^{mn} \, \Phi_{mn} \,] \, E^{-1} \\
(\gamma^2 \Psi^* \otimes \gamma^2 \Psi^*) &= [\, \gamma^n \, \Phi^*_n + i \, \sigma^{mn} \, \Phi^*_{mn} \,] \, E^{-1} \, .
\end{aligned}
\tag{265}
$$

Complex vectors and tensors are given by (see (245))

$$
\begin{aligned}
A_n + iB_n = \Phi_n &= \frac{1}{4} \, \text{Sp} \, [\, E\gamma_n (\Psi \otimes \Psi) \,] \, , \\
A_n - iB_n = \Phi^*_n &= \frac{1}{4} \, \text{Sp} \, [\, E\gamma_n (\gamma^2 \Psi^* \otimes \gamma^2 \Psi^*) \,] \, , \\
\Phi_{mn} &= -\frac{1}{2i} \, \text{Sp} \, [\, E\sigma_{mn} (\Psi \otimes \Psi) \,] \, , \\
\Phi^*_{mn} &= -\frac{1}{2i} \, \text{Sp} \, [\, E\sigma_{mn} (\gamma^2 \Psi^* \otimes \gamma^2 \Psi^*) \,] \, .
\end{aligned}
\tag{266}
$$

We may use conventional notation $\gamma^2 \Psi^* = \Psi^c$, then the formulas look shorter

$$
\begin{aligned}
A_n &= \frac{1}{8} \, \text{Sp} \, [\, E\gamma_n (\Psi \otimes \Psi + \Psi^c \otimes \Psi^c) \,] \, , \\
iB_n &= \frac{1}{8} \, \text{Sp} \, [\, E\gamma_n (\Psi^c \otimes \Psi^c - \Psi^c \otimes \Psi^c) \,] \, , \\
\Phi_{mn} &= -\frac{1}{2i} \, \text{Sp} \, [\, E\sigma_{mn} (\Psi \otimes \Psi) \,] \, , \\
\Phi^*_{mn} &= -\frac{1}{2i} \, \text{Sp} \, [\, E\sigma_{mn} (\Psi^c \otimes \Psi^c) \,] \, .
\end{aligned}
\tag{267}
$$

Let specify the complex tensor

$$
\begin{aligned}
\Phi^{01} &= \frac{i}{4} \, [\, (\xi^1 \xi^1 - \xi^2 \xi^2) + (\eta_{\dot{1}} \eta_{\dot{1}} - \eta_{\dot{2}} \eta_{\dot{2}}) \,] \, , \\
\Phi^{23} &= \frac{1}{4} \, [\, (\xi^1 \xi^1 - \xi^2 \xi^2) - (\eta_{\dot{1}} \eta_{\dot{1}} - \eta_{\dot{2}} \eta_{\dot{2}}) \,] \, , \\
s^1 = \Phi^{01} + i\Phi^{23} &= \frac{i}{2} \, (\xi^1 \xi^1 - \xi^2 \xi^2) \, , \\
t^1 = \Phi^{01} - i\Phi^{23} &= \frac{i}{2} \, (\eta_{\dot{1}} \eta_{\dot{1}} - \eta_{\dot{2}} \eta_{\dot{2}}) \, ;
\end{aligned}
$$

$$
\begin{aligned}
\Phi^{02} &= -\frac{1}{4} \, [\, (\xi^1 \xi^1 + \xi^2 \xi^2) + (\eta_{\dot{1}} \eta_{\dot{1}} + \eta_{\dot{2}} \eta_{\dot{2}}) \,] \, , \\
\Phi^{31} &= -\frac{1}{4i} \, [\, (\xi^1 \xi^1 + \xi^2 \xi^2) - (\eta_{\dot{1}} \eta_{\dot{1}} + \eta_{\dot{2}} \eta_{\dot{2}}) \,] \, , \\
s^2 = \Phi^{02} + i\Phi^{31} &= -\frac{1}{2} \, (\xi^1 \xi^1 + \xi^2 \xi^2) \, , \\
t^2 = \Phi^{02} - i\Phi^{31} &= -\frac{1}{2} \, (\eta_{\dot{1}} \eta_{\dot{1}} + \eta_{\dot{2}} \eta_{\dot{2}}) \, ;
\end{aligned}
$$

$$\Phi^{03} = -\frac{i}{2} \; (\; \xi^1 \xi^2 + \eta_{\dot 1} \eta_{\dot 2}) \; ,$$
$$\Phi^{12} = -\frac{1}{2} \; (\; \xi^1 \xi^2 - \eta_{\dot 1} \eta_{\dot 2}) \; ,$$
$$s^3 = \Phi^{03} + i \Phi^{12} = -i \; \xi^1 \xi^2 \; ,$$
$$t^3 = \Phi^{03} - i \Phi^{12} = -i \; \eta_{\dot 1} \eta_{\dot 2} \; . \qquad (268)$$

The vectors $\mathbf{s}$ and $\mathbf{t}$ are isotropic:

$$\mathbf{s}^2 = -\frac{1}{4} \; (\xi^1 \xi^1 - \xi^2 \xi^2)^2 + \frac{1}{4} \; (\xi^1 \xi^1 + \xi^2 \xi^2)^2 - (\xi^1 \xi^2)^2 \equiv 0 \; .$$
$$\mathbf{t}^2 = -\frac{1}{4} \; (\eta_{\dot 1} \eta_{\dot 1} - \eta_{\dot 2} \eta_{\dot 2})^2 + \frac{1}{4} \; (\eta_{\dot 1} \eta_{\dot 1} + \eta_{\dot 2} \eta_{\dot 2})^2 - (\eta_{\dot 1} \eta_{\dot 2})^2 \equiv 0 \; .$$

besides

$$\mathbf{s} \; \mathbf{t} = -\frac{1}{4} \; (\xi^1 \xi^1 - \xi^2 \xi^2)(\eta_{\dot 1} \eta_{\dot 1} - \eta_{\dot 2} \eta_{\dot 2}) +$$
$$+ \frac{1}{4} \; (\xi^1 \xi^1 + \xi^2 \xi^2)(\eta_{\dot 1} \eta_{\dot 1} + \eta_{\dot 2} \eta_{\dot 2}) - \xi^1 \xi^2 \; \eta_{\dot 1} \eta_{\dot 2} = \frac{1}{2} (\xi^1 \eta_{\dot 2} - \xi^2 \eta_{\dot 1})^2 \; . \qquad (269)$$

Let us check the sign of the relativistic length for A_n (it equal that for B_n):

$$2A_0 = (\xi^1 \eta_{\dot 2} - \xi^2 \eta_{\dot 1}) + (\xi^{1*} \eta^*_{\dot 2} - \xi^{2*} \eta^*_{\dot 1}) \; ,$$
$$2A_3 = - \; (\xi^1 \eta_{\dot 2} + \xi^2 \eta_{\dot 1}) - (\xi^{1*} \eta^*_{\dot 2} + \xi^{2*} \eta^*_{\dot 1}) \; ,$$
$$A_1 = (\xi^1 \eta_{\dot 1} - \xi^2 \eta_{\dot 2}) + (\xi^{1*} \eta^*_{\dot 1} - \xi^{2*} \eta^*_{\dot 2}) \; ,$$
$$A_2 = i \; (\xi^1 \eta_{\dot 1} + \xi^2 \eta_{\dot 2}) - i(\xi^{1*} \eta^*_{\dot 1} + \xi^{2*} \eta^*_{\dot 2}) \; ;$$

allowing for identities

$$4(A_0^2 - A_3^2) = (\xi^1 \eta_{\dot 2} - \xi^2 \eta_{\dot 1})^2 - 2(\xi^1 \eta_{\dot 2} - \xi^2 \eta_{\dot 1})(\xi^{1*} \eta^*_{\dot 2} - \xi^{2*} \eta^*_{\dot 1}) + (\xi^{1*} \eta^*_{\dot 2} - \xi^{2*} \eta^*_{\dot 1})^2 -$$
$$- (\xi^1 \eta_{\dot 2} + \xi^2 \eta_{\dot 1})^2 - 2(\xi^1 \eta_{\dot 2} + \xi^2 \eta_{\dot 1})(\xi^{*1} \eta^*_{\dot 2} + \xi^{2*} \eta^*_{\dot 1}) - (\xi^{1*} \eta^*_{\dot 2} + \xi^{2*} \eta^*_{\dot 1})^2 =$$
$$= -4 \xi^1 \xi^2 \; \eta_{\dot 1} \eta_{\dot 2} - 4 \xi^{1*} \xi^{2*} \; \eta^*_{\dot 1} \eta^*_{\dot 2} - 4 \xi^1 \xi^{1*} \; \eta_{\dot 2} \eta^*_{\dot 2} - 4 \xi^2 \xi^{2*} \; \eta_{\dot 1} \eta^*_{\dot 1} \; ,$$

$$4(A_1^2 + A_2^2) = (\xi^1 \eta_{\dot 1} - \xi^2 \eta_{\dot 2})^2 + 2(\xi^1 \eta_{\dot 1} - \xi^2 \eta_{\dot 2})(\xi^{1*} \eta^*_{\dot 1} - \xi^{2*} \eta^*_{\dot 2}) + (\xi^{1*} \eta^*_{\dot 1} - \xi^{2*} \eta^*_{\dot 2})^2 -$$
$$- (\xi^1 \eta_{\dot 1} + \xi^2 \eta_{\dot 2})^2 + 2(\xi^1 \eta_{\dot 1} + \xi^2 \eta_{\dot 2})(\xi^{1*} \eta^*_{\dot 1} + \xi^{2*} \eta^*_{\dot 2}) - (\xi^{1*} \eta^*_{\dot 1} + \xi^{2*} \eta^*_{\dot 2})^2 =$$
$$= -4 \xi^1 \xi^2 \; \eta_{\dot 1} \eta_{\dot 2} - 4 \xi^{1*} \xi^{2*} \; \eta^*_{\dot 1} \eta^*_{\dot 2} + 4 \xi^1 \xi^{1*} \; \eta_{\dot 1} \eta^*_{\dot 1} + 4 \xi^2 \xi^{2*} \; \eta_{\dot 2} \eta^*_{\dot 2} \; ,$$

we arrive at

$$A_0^2 - \mathbf{A}^2 =$$
$$= -\xi^1 \xi^2 \; \eta_{\dot 1} \eta_{\dot 2} - \xi^{1*} \xi^{2*} \; \eta^*_{\dot 1} \eta^*_{\dot 2} - \xi^1 \xi^{1*} \; \eta_{\dot 2} \eta^*_{\dot 2} - \xi^2 \xi^{2*} \; \eta_{\dot 1} \eta^*_{\dot 1} +$$
$$+ \xi^1 \xi^2 \; \eta_{\dot 1} \eta_{\dot 2} + \xi^{1*} \xi^{2*} \; \eta^*_{\dot 1} \eta^*_{\dot 2} - \xi^1 \xi^{1*} \; \eta_{\dot 1} \eta^*_{\dot 1} - \xi^2 \xi^{2*} \; \eta_{\dot 2} \eta^*_{\dot 2} =$$
$$= -\xi^1 \xi^{1*} \; \eta_{\dot 2} \eta^*_{\dot 2} - \xi^2 \xi^{2*} \; \eta_{\dot 1} \eta^*_{\dot 1} - \xi^1 \xi^{1*} \; \eta_{\dot 1} \eta^*_{\dot 1} - \xi^2 \xi^{2*} \; \eta_{\dot 2} \eta^*_{\dot 2} =$$
$$= -(\xi^1 \xi^{1*} + \xi^2 \xi^{2*}) \; (\eta_{\dot 1} \eta^*_{\dot 1} + \eta_{\dot 2} \eta^*_{\dot 2}) < 0 \; . \qquad (270)$$

Therefore, this 4-vector is space-like, and it cannot correspond to a time-like Stoke 4-vector.

17 Spinor representation for a time-like vectors, on possible Jones spinor for a partly polarized light

Now let us examine else one possibility

$$\Psi \otimes (-i\Psi^c) = (\xi^1, \xi^2, \eta_{\dot{1}}, \eta_{\dot{2}}) \otimes \left| \begin{array}{c} +\eta^*_{\dot{2}} \\ -\eta^*_{\dot{1}} \\ -\xi^{2*} \\ +\xi^{1*} \end{array} \right| =$$

$$= \left| \begin{array}{cccc} +\xi^1\eta^*_{\dot{2}} & -\xi^1\eta^*_{\dot{1}} & -\xi^1\xi^{2*} & +\xi^1\xi^{1*} \\ +\xi^2\eta^*_{\dot{2}} & -\xi^2\eta^*_{\dot{1}} & -\xi^2\xi^{2*} & +\xi^2\xi^{1*} \\ +\eta_{\dot{1}}\eta^*_{\dot{2}} & -\eta_{\dot{1}}\eta^*_{\dot{1}} & -\eta_{\dot{1}}\xi^{2*} & +\eta_{\dot{1}}\xi^{1*} \\ +\eta_{\dot{2}}\eta^*_{\dot{2}} & -\eta_{\dot{2}}\eta^*_{\dot{1}} & -\eta_{\dot{2}}\xi^{2*} & +\eta_{\dot{2}}\xi^{1*} \end{array} \right| = \left| \begin{array}{cccc} \xi^{11} & \xi^{12} & \Delta^1{}_{\dot{1}} & \Delta^1{}_{\dot{2}} \\ \xi^{21} & \xi^{22} & \Delta^2{}_{\dot{1}} & \Delta^2{}_{\dot{2}} \\ H_{\dot{1}}{}^1 & H_{\dot{1}}{}^2 & \eta_{\dot{1}\dot{1}} & \eta_{\dot{1}\dot{2}} \\ H_{\dot{2}}{}^1 & H_{\dot{2}}{}^2 & \eta_{\dot{2}\dot{1}} & \eta_{\dot{2}\dot{2}} \end{array} \right| . \tag{271}$$

Corresponding 4-vector is determined by

$$\Phi_0 = \frac{1}{2} \, [\, (H_{\dot{2}}{}^1 - H_{\dot{1}}{}^2) - (\Delta^2{}_{\dot{1}} - \Delta^1{}_{\dot{2}}) \,] = \frac{1}{2} \, [\, (\eta_{\dot{2}}\eta^*_{\dot{2}} + \eta_{\dot{1}}\eta^*_{\dot{1}}) + (\xi^2\xi^{2*} + \xi^1\xi^{1*}) \,] > 0 \, ,$$

$$\Phi_3 = -\frac{1}{2} \, [\, (H_{\dot{2}}{}^1 + H_{\dot{1}}{}^2) + (\Delta^2{}_{\dot{1}} + \Delta^1{}_{\dot{2}}) \,] = -\frac{1}{2} \, [\, (\eta_{\dot{2}}\eta^*_{\dot{2}} - \eta_{\dot{1}}\eta^*_{\dot{1}}) + (-\xi^2\xi^{2*} + \xi^1\xi^{1*}) \,] \, ,$$

$$\Phi_1 = \frac{1}{2} \, [\, (H_{\dot{1}}{}^1 - H_{\dot{2}}{}^2) + (\Delta^1{}_{\dot{1}} - \Delta^2{}_{\dot{2}}) \,] = \frac{1}{2} \, [\, (\eta_{\dot{1}}\eta^*_{\dot{2}} + \eta_{\dot{2}}\eta^*_{\dot{1}}) - (\xi^1\xi^{2*} + \xi^2\xi^{1*}) \,] \, ,$$

$$\Phi_2 = \frac{i}{2} \, [\, (H_{\dot{1}}{}^1 + H_{\dot{2}}{}^2) + (\Delta^1{}_{\dot{1}} + \Delta^2{}_{\dot{2}}) \,] = \frac{i}{2} \, [(\eta_{\dot{1}}\eta^*_{\dot{2}} - \eta_{\dot{2}}\eta^*_{\dot{1}}) + (-\xi^1\xi^{2*} + \xi^2\xi^{1*}) \,] \, .$$

Allowing for

$$4(\Phi_0^2 - \Phi_3^2) = (\eta_{\dot{2}}\eta^*_{\dot{2}} + \eta_{\dot{1}}\eta^*_{\dot{1}})^2 + 2(\eta_{\dot{2}}\eta^*_{\dot{2}} + \eta_{\dot{1}}\eta^*_{\dot{1}})(\xi^2\xi^{2*} + \xi^1\xi^{1*}) + (\xi^2\xi^{2*} + \xi^1\xi^{1*})^2 -$$
$$-(\eta_{\dot{2}}\eta^*_{\dot{2}} - \eta_{\dot{1}}\eta^*_{\dot{1}})^2 - 2(\eta_{\dot{2}}\eta^*_{\dot{2}} - \eta_{\dot{1}}\eta^*_{\dot{1}})(-\xi^2\xi^{2*} + \xi^1\xi^{1*}) + (-\xi^2\xi^{2*} + \xi^1\xi^{1*})^2 =$$
$$= 4\eta_{\dot{1}}\eta^*_{\dot{1}} \, \eta_{\dot{2}}\eta^*_{\dot{2}} + 4\xi^1\xi^{1*} \, \xi^2\xi^{2*} + 4\eta_{\dot{1}}\eta^*_{\dot{1}} \, \xi^1\xi^{1*} + 4\eta_{\dot{2}}\eta^*_{\dot{2}} \, \xi^2\xi^{2*} \, ,$$
$$4(\Phi_1^2 + \Phi_2^2) = (\eta_{\dot{1}}\eta^*_{\dot{2}} + \eta_{\dot{2}}\eta^*_{\dot{1}})^2 - 2(\eta_{\dot{1}}\eta^*_{\dot{2}} + \eta_{\dot{2}}\eta^*_{\dot{1}})(\xi^1\xi^{2*} + \xi^2\xi^{1*}) + (\xi^1\xi^{2*} + \xi^2\xi^{1*})^2 -$$
$$-(\eta_{\dot{1}}\eta^*_{\dot{2}} - \eta_{\dot{2}}\eta^*_{\dot{1}})^2 - 2(\eta_{\dot{1}}\eta^*_{\dot{2}} - \eta_{\dot{2}}\eta^*_{\dot{1}})(-\xi^1\xi^{2*} + \xi^2\xi^{1*}) - (-\xi^1\xi^{2*} + \xi^2\xi^{1*})^2 =$$
$$= 4\eta_{\dot{1}}\eta^*_{\dot{1}} \, \eta_{\dot{2}}\eta^*_{\dot{2}} \; + 4\xi^1\xi^{1*} \, \xi^2\xi^{2*} - 4\eta_{\dot{1}}\eta^*_{\dot{2}} \, \xi^2\xi^{1*} - 4\eta_{\dot{2}}\eta^*_{\dot{1}}\xi^1\xi^{2*} \, ,$$

So that
we get

$$\Phi^a\Phi_a = \Phi_0^2 - \Phi_1^2 - \Phi_2^2 - \Phi_1^3 =$$
$$= \eta_{\dot{1}}\eta^*_{\dot{1}} \, \xi^1\xi^{1*} + \eta_{\dot{2}}\eta^*_{\dot{2}} \, \xi^2\xi^{2*} + \eta_{\dot{1}}\eta^*_{\dot{2}} \, \xi^2\xi^{1*} + \eta_{\dot{2}}\eta^*_{\dot{1}}\xi^1\xi^{2*} \, . \tag{272}$$

Let us demonstrate that this vector is time-like. With the notation[5]

$$\xi = \left| \begin{array}{c} N_1 e^{in_1} \\ N_2 e^{in_2} \end{array} \right| , \qquad \eta = \left| \begin{array}{c} M_1 e^{im_1} \\ M_2 e^{im_2} \end{array} \right| , \tag{273}$$

[5]It should be noted that here 8 real parameters are introduced.

we get

$$\Phi^a \Phi_a = N_1^2 M_1^2 + N_2^2 M_2^2 + 2N_1 M_1 \; N_2 M_2 \; \cos\,[(n_1 - n_2) - (m_1 - m_2)] \; ;$$

therefore

$$(N_1 M_1 - N_2 M_2)^2 < \Phi_0^2 - \Phi_1^2 - \Phi_2^2 - \Phi_1^3 < (N_1 M_1 + N_2 M_2)^2 \; . \tag{274}$$

This means that we have ground to consider 4-vector Φ_a as Stokes 4-vector S_a:

$$(N_1 M_1 - N_2 M_2)^2 < S_0^2 - \mathbf{S}^2 < (N_1 M_1 + N_2 M_2)^2 \; , \tag{275}$$

and two 2-spinors (273) as making up a Jones bi-spinor corresponding a partly polarized light.

It remains to find explicit form for corresponding (real) Stokes 4-tensor S_{ab}:

$$\begin{aligned}
\Phi^{01} &= \frac{i}{4} \; [\; (\xi^1 \eta^*_{\dot 2} + \xi^2 \eta^*_{\dot 1}) - (\eta_{\dot 1} \xi^{2*} + \eta_{\dot 2} \xi^{1*}) \;] \; , \\
\Phi^{23} &= \frac{1}{4} \; [\; (\xi^1 \eta^*_{\dot 2} + \xi^2 \eta^*_{\dot 1}) + (\eta_{\dot 1} \xi^{2*} + \eta_{\dot 2} \xi^{1*}) \;] \; , \\
\Phi^{02} &= -\frac{1}{4} \; [\; (\xi^1 \eta^*_{\dot 2} - \xi^2 \eta^*_{\dot 1}) + (-\eta_{\dot 1} \xi^{2*} + \eta_{\dot 2} \xi^{1*}) \;] \; , \\
\Phi^{31} &= \frac{i}{4} \; [\; (\xi^1 \eta^*_{\dot 2} - \xi^2 \eta^*_{\dot 1}) - (-\eta_{\dot 1} \xi^{2*} + \eta_{\dot 2} \xi^{1*}) \;] \; , \\
\Phi^{03} &= -\frac{i}{4} \; [\; (\xi^2 \eta^*_{\dot 2} - \xi^1 \eta^*_{\dot 1}) + (-\eta_{\dot 2} \xi^{2*} + \eta_{\dot 1} \xi^{1*})] \; , \\
\Phi^{12} &= -\frac{1}{4} \; [\; (\xi^2 \eta^*_{\dot 2} - \xi^1 \eta^*_{\dot 1}) - (-\eta_{\dot 2} \xi^{2*} + \eta_{\dot 1} \xi^{1*})] \; .
\end{aligned} \tag{276}$$

and

$$\begin{aligned}
s^1 = a^1 + i b^1 &= \frac{i}{2} \; (\xi^1 \eta^*_{\dot 2} + \xi^2 \eta^*_{\dot 1}) \; , \\
s^2 = a^2 + i b^2 &= -\frac{1}{2} \; (\xi^1 \eta^*_{\dot 2} - \xi^2 \eta^*_{\dot 1}) \; , \\
s^3 = a^3 + i b^3 &= -\frac{i}{2} \; (\xi^2 \eta^*_{\dot 2} - \xi^1 \eta^*_{\dot 1}) \; ;
\end{aligned} \tag{277}$$

besides this complex 3-vector is not isotropic:

$$\mathbf{s}^2 = -\frac{1}{4} \; (\xi^1 \eta^*_{\dot 1} - \xi^2 \eta^*_{\dot 1})^2 \neq 0 \; .$$

Last remark should be added. Results of this section can be of use not only in polarization optics, but also they may be of interest to describe Maxwell theory in spinor approach, when instead variables A_n, F_{mn} one introduces one fundamental electromagnetic bi-spinors $\Psi = (\xi, \eta)$. Also, they could have meaning in the context of explicit constructing models for space-time with spinor structure.

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